On equicontinuity of mappings with branching defined in family of domains


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All definitions and notions used below may be found in [1]. Let $S_i=S(x_0,r_i)=\{x\in\mathbb{R}^n: |x-x_0|=r_i\}$, $i=1,2$, $n\geqslant 2$, and let $Q:\mathbb{R}^n\rightarrow\mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x)\equiv 0$ for $x\in\mathbb{R}^n\setminus D$. In what follows, $M_p(\cdot)$ denotes the modulus of family of paths of the order $p\geqslant 1$, and $dm(x)$ is an element of the Lebesgue measure in $\mathbb{R}^n$. Let $A=A(x_0,r_1,r_2)=\{x\in\mathbb{R}^n: r_1<|x-x_0|<r_2\}$. Given $p\geqslant 1$, a mapping $f:D\rightarrow\overline{\mathbb{R}^n}$ is called a *ring $Q$-mapping at a point $x_0\in\mathbb{R}^n$ with respect to $p$-modulus*, if the condition

$$M_p(f(\Gamma(S_1,S_2,D)))\leqslant\int\limits_{A\cap D} Q(x)\cdot\eta^p(|x-x_0|)\,dm(x) \qquad (1)$$

holds for all $0<r_1<r_2<\infty$ and any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \qquad (2)$$

Below $h(x,y)$ denotes the chordal (spherical metric) between points $x,y\in\overline{\mathbb{R}^n}$. Let $h(E)$ be achordal diameter of the set $E$ in $\overline{\mathbb{R}^n}$. For $x_0\in\mathbb{R}^n$ and $r_0>0$, as usual, $B(x_0,r_0)=\{x\in\mathbb{R}^n: |x-x_0|<r_0\}$. Given $p\geqslant 1$, a domain $D_0\subset\mathbb{R}^n$, a set $E\subset\overline{\mathbb{R}^n}$ and a number $\delta>0$, a fixed domain $D_0\subset\mathbb{R}^n$, $n\geqslant 2$, a sequence of domains $\mathfrak{D}=\{D_m\}_{m=1}^{\infty}$ the kernel of which is $D_0$ and a Lebesgue measurable function $Q:\mathbb{R}^n\rightarrow[0,\infty]$ we denote by $\mathfrak{R}_{Q,\delta,p,E}(D_0,\mathfrak{D})$ the family of all open discrete closed mappings $f:D_m\rightarrow\overline{\mathbb{R}^n}\setminus E$ satisfying (1)–(2) for any $0<r_1<r_2<\infty$ at any point $x_0\in\overline{D_0}$ such that the following condition holds: for any domain $D_m^{\prime}:=f_m(D_m)$ there exists a continuum $K_m\subset D_m^{\prime}$ such that $h(K_m)\geqslant\delta$ and $h(f_m^{-1}(K_m),\partial D_m)\geqslant\delta>0$. The following statement holds.

**Theorem 1.** *Let $p\in(n-1,n]$ and let $f_m\in\mathfrak{R}_{Q,\delta,p,E}(D_0,\mathfrak{D})$, $m=1,2,\ldots$, be a sequence such that:*

*1) the sequence of domains $D_m$ is regular with respect to $D_0$;*

*2) for every $m\in\mathbb{N}$, a domain $D_m$ is locally connected on its boundary;*

*3) the family $f_m(D_m)$ is equi-uniform with respect to $p$-modulus over all $m\in\mathbb{N}$;*

*4) at least one of two following conditions hold: $Q$ has a finite mean oscillation in $\overline{D_0}$, or*

$$\int\limits_{0}^{\beta(x_0)}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\prime\,\frac{1}{p-1}}(t)}=\infty \qquad (3)$$

*for every* $x_0 \in \overline{D_0}$ *and some* $\beta(x_0) > 0$, *where* $q'_{x_0}(t)$ *denotes the integral average of the function* $Q'$ *under the sphere* $S(x_0, t)$, *while* $Q'(x) = \begin{cases} Q(x), & Q(x) \geqslant 1\,, \\ 1, & Q(x) < 1\,. \end{cases}$ *Let* $E$ *be a set of a positive capacity for* $p = n$, *and* $E$ *is arbitrary closed set for* $n - 1 < p < n$.

***I.*** *Then the family* $f_m$, $m = 1, 2, \ldots$, *is uniformly equicontinuous in* $\mathfrak{D}$, *i.e., for any* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that* $h(f_m(x), f_m(y)) < \varepsilon$ *whenever* $x, y \in D_m$, $|x - y| < \delta$ *and* $m \in \mathbb{N}$. *Moreover, there is a subsequence* $f_{m_k}$, $k = 1, 2, \ldots$, *which converges to* $f$ *locally uniformly in* $D_0$. *In this case,* $f$ *has a continuous boundary extension* $f : \overline{D_0} \to \overline{\mathbb{R}^n}$. *If* $x_m \in D_m$, $m = 1, 2, \ldots$, $f_m$ *converges to* $f$ *locally uniformly in* $D_0$ *and* $x_m \to x_0$ *as* $m \to \infty$, *then* $f_m(x_m) \to f(x_0)$.

***II.*** *In addition, assume that*

*5)* $p = n$ *and there is* $r > 0$, *does not depending on* $m \in \mathbb{N}$, *such that* $h(E) \geqslant r$ *whenever* $E$ *is any component of* $\partial f_m(D_m)$. *If* $A_0$ *is some compactum in* $D_0$, *then there is* $\delta_2 > 0$ *and* $M_0 \in \mathbb{N}$ *such that* $A_0 \subset D_m$ *and* $h(f_m(A_0), \partial f_m(D_m)) \geqslant \delta_2$ *for every* $m \geqslant M_0$. *The mapping* $f$ *is boundary preserving: if* $x_0 \in \partial D_0$, *then* $f(x_0) \in \partial f(D_0)$. *If* $\overline{B(x_0, \varepsilon_0)} \subset D_0 \cap \bigcap\limits_{m=1}^{\infty} D_m$, *then there is* $\varepsilon_1 > 0$ *does not depending on* $m \in \mathbb{N}$ *such that*

$$B_h(f_m(x_0), \varepsilon_1) \subset f_m(D_m)\,, \quad m = 1, 2, \ldots\,, \tag{4}$$

*where* $B_h(y_0, r_0) = \{y \in \overline{\mathbb{R}^n} : h(y, y_0) < r_0\}$.

The mentioned result is obtained in [2].

## References

[1] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory.* Springer Science + Business Media, LLC : New York, 2009.

[2] Ilkevych N., Romash D., Sevost'yanov E. On distortion estimates under mappings of families of domains. https://arxiv.org/abs/2602.14013 .

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