

# FROM STARS TO DOUBLE-STARS VIA ULTRAMETRICS

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By labeled tree  $T(l)$  we mean a tree  $T$  equipped with a labeling  $l: V(T) \rightarrow \mathbb{R}^+$ , where  $V(T)$  is the vertex set of  $T$ . Let  $T(l)$  be a labeled tree. As in [1] we define a mapping  $d_l: V(T) \times V(T) \rightarrow \mathbb{R}^+$  by

$$d_l(u, v) = \begin{cases} 0, & \text{if } u = v, \\ \max_{w \in V(P)} l(w), & \text{otherwise,} \end{cases}$$

where  $P$  is the path joining  $u$  and  $v$  in  $T$ . Let  $(Y, \rho)$  be an ultrametric space. We say that  $(Y, \rho)$  is generated by  $l: V(T) \rightarrow \mathbb{R}^+$  if the equality  $(Y, \rho) = (V(T), d_l)$  holds.

The concept of ultrametric spaces generated by non-negative vertex labelings on both finite and infinite trees was introduced in [1] and investigated in [3, 4, 7, 9]. Recently, in [5], the centers of distances of ultrametric spaces generated by labeled trees were described. The ultrametric spaces generated by vertex labeled star graphs are studied in [2, 8, 9, 6]. Here we would like to present a solution of the following problem.

**Problem 1.** Describe the structure of trees  $T$  which satisfy the following condition: If  $d$  is an arbitrary ultrametric generated by vertex labeling of  $T$ , then the ultrametric space  $(V(T), d)$  is isometric to an ultrametric space generated by vertex labeling of some star graph.

The following theorem gives a solution of Problem 1.

**Theorem 2.** Let  $T$  be a tree,  $U(T)$  be the set of all ultrametric spaces generated by vertex labelings of  $T$ , and let  $\mathbf{US}$  be the class of all ultrametric spaces generated by labelings of star graphs. Then the following statements are equivalent:

- (i) The inclusion  $U(T) \subseteq \mathbf{US}$  holds.
- (ii) The length of the longest path in  $T$  does not exceed three.

The above theorem implies the following corollary.

**Corollary 3.** Let  $T$  be a tree. Then the inclusion  $U(T) \subseteq \mathbf{US}$  holds if and only if  $T$  is isomorphic either to a star-graph or to a double-star graph.

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