On analogue of Koebe-Bloch theorem for ring homeomorphisms

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All definitions and notions used below may be found in [1]. Let $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, $i = 1, 2$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. In what follows, $M(\cdot)$ denotes the modulus of family of paths, and $dm(x)$ is an element of the Lebesgue measure in $\mathbb{R}^n$. A mapping $f : D \to \overline{\mathbb{R}^n}$, $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$, is called a *ring Q-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$*, if the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A \cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x)\,,$$

$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, holds for all $0 < r_1 < r_2 < d_0 := \sup\limits_{x \in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,.$$

Below $h(x, y)$ denotes the chordal (spherical metric) between points $x, y \in \overline{\mathbb{R}^n}$. Let $h(E)$ be achordal diameter of the set E in $\overline{\mathbb{R}^n}$. Given $x_0 \in \overline{\mathbb{R}^n}$ and $r > 0$ we set $B_h(x_0, r) := \{x \in \overline{\mathbb{R}^n} : h(x, x_0) < r\}$. For $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, as usual, $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$. A family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{R}^n}$ is called *uniformly open at a point* $x_0 \in D$, if for every $\varepsilon_0 > 0$, $\varepsilon_0 < \mathrm{dist}\,(x_0, \partial D)$, there is $r_0 = r_0(x_0, \varepsilon_0) > 0$ such that $B_h(f(x_0), r_0) \subset f(B(x_0, \varepsilon_0))$ for every $f \in \mathfrak{F}$. A family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{R}^n}$ is called *uniformly open on* K, if for every $\varepsilon_0 > 0$ there exists $r_0 = r_0(K, \varepsilon_0) > 0$ such that $B_h(f(x_0), r_0) \subset f(B(x_0, \varepsilon_0))$ for every $f \in \mathfrak{F}$ and every $B(x_0, \varepsilon_0) \subset K$.

Given a domain D in $\mathbb{R}^n$, $n \geqslant 2$, a Lebesgue measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and a number $\delta > 0$ denote by $\mathfrak{F}^{\delta}_{K,Q}(D)$ a family of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ such that $h(f(K), \partial f(D)) = \inf\limits_{x \in f(K), y \in \partial f(D)} h(x, y) \geqslant \delta$. The following result holds.

Theorem 1. (An analogue of Koebe-Bloch theorem). *Let D be a bounded domain. Assume that, for each point $x_0 \in D$ and for every $0 < r_1 < r_2 < r_0 := \sup\limits_{x \in D} |x - x_0|$ there is a set $E_1 \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function Q is integrable with respect to $\mathcal{H}^{n-1}$ over the spheres $S(x_0, r)$ for every $r \in E_1$. Then the family $\mathfrak{F}^{\delta}_{K,Q}(D)$ is uniformly open on K.*

Remark 2. If f is a homeomorphism, $f : D \to \mathbb{R}^n$, $n \geqslant 3$, $f \in W^{1,\varphi}_{\mathrm{loc}}$, $\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty$ and $K_O(x, f) \in L^{n-1}_{\mathrm{loc}}$, then $g := f^{-1} \in W^{1,n}_{\mathrm{loc}}$, where $K_O(x, f)$ is an outher dilatation of f at x. On the other hand, if $K_I(y, g) \in L^1_{\mathrm{loc}}$, then g is a ring Q_*-homeomorphism with $Q_* = K_I(y, g)$ (see

Corollary 8.5, Theorem 8.6 in [1]). Thus, Theorem 1 holds for Orlicz-Sobolev classes $W^{1,\varphi}_{\rm loc}$ under above assumptions.

References

[1] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory.* Springer Science + Business Media, LLC : New York, 2009.

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