Subwreath product as structure of normal subgroups of symmetric group wreath product

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In this research, we continue our previous investigation of wreath product normal structure [7, 8]. Normal subgroups and their structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in the decomposition of a permutation π of degree n.

The number $n - k(\pi)$ is denoted by $\mathrm{dec}(\pi)$, and is called a decrement [6] of permutation π. As is well known [6], the minimal number of transpositions in a factorization of a permutation π into transpositions happens to be equal to $\mathrm{dec}(\pi)$. We set $\mathrm{dec}(e) = 0$. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$\mathrm{dec}(\pi_1 \cdot \pi_2) = \mathrm{dec}(\pi_1) + \mathrm{dec}(\pi_2) - 2m, \quad m \in \mathbf{N}. \tag{1}$$

Definition 1. The permutational *subwreath product* $G \wr\!\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [1] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n$, $n \geqslant 3$, which is presented by the Kaloujnine tableaux [4] of the form $[e]_0$, $[a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} \mathrm{dec}([a_i]_1) = 2k, \quad k \in \mathbb{N}, \tag{2}$$

is denoted by $\widetilde{A}_n^{(1)}$. Note that condition (2) uniquely identifies the subdirect product.

We extend the definition of $\widetilde{A}_n^{(1)}$ to the 3-multiple wreath product in a recursive way.

We also prove that $\widetilde{A}_n^{(1)}$ is the **normal subgroup** of $S_n \wr S_n$, so it will be called the ***alternating level subgroup*** $\widetilde{A}_n^{(1)}$ with structure $E \wr\!\wr \widetilde{A}_n$, where $\widetilde{A}_n$ is the subdirect product of n copies of S_n uniquely identified by (2).

Theorem 3. *The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** $n-1$ in $S_n \wr S_n$, $n \geqslant 3$, $n \cong 1 \pmod 2$, and **normal rank** n iff $n \geqslant 3$, $n \cong 0 \pmod 2$.*

Definition 4. *The subgroup $E \wr \widetilde{A}_n^{(1)}$ is denoted by $\widetilde{A}_n^{(2)}$.*

Furthermore, we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** 2 [2] in $S_n \wr S_n$.

Definition 5. The set of elements from $S_n \wr S_n \wr S_n$, $n \geqslant 3$, presented by the tables [4] of the form $[e]_0$, $[e, e, \ldots, e]_1$, $[a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} \mathrm{dec}([a_i]_2) = 2k, \quad k \in \mathbb{N}, \tag{3}$$

is denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies the subdirect product in $\prod_{i=1}^{n^2} S_n$ as the base of the subwreath product; the similar subdirect product describing the commutator of a wreath product was investigated by us in [9], and in research of pronormality it appears in [5].

Proposition 6. *The subgroup $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$, as well as $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. Furthermore, $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.*

Definition 7. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

(1) *elements of $E \wr A_n$,*
(2) *elements with the tableau [4] presentation $[e]_1$, $[\pi_1, \ldots, \pi_n]_2$, such that $\pi_i \in S_n \setminus A_n$.*

One can easily verify the correctness of this definition, i.e., that the set of such elements forms a subgroup, and its normality. This subgroup has structure

$$\tilde{T_n} \simeq \left(\underbrace{A_n \times A_n \times \cdots \times A_n}_{n} \right) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ of a subdirect product is subject to items 1) and 2).

Remark 8. The order of $\widetilde{T_n}$ is $\dfrac{(n!)^n}{2^{n-1}}$.

Definition 9. The unique minimal normal subgroup is called the monolith.

Theorem 10. *The monolith of $S_n \wr S_m$ is $e \wr A_m$.*

Theorem 11. *Proper normal subgroups in $S_n \wr S_m$, where $n, m \geq 3$ with $n, m \neq 4$, are of the following types:*

(1) *subgroups that act only on the second level are*

$$E \wr\wr \widetilde{A}_m,\ \widetilde{T}_m,\ E \wr S_m,\ E \wr A_m;$$

(2) *subgroups that act on both levels are $A_n \wr\wr \widetilde{A}_m$, $S_n \wr\wr \widetilde{A}_m$, $A_n \wr S_m$,*

wherein the subgroup $S_n \wr\wr \widetilde{A}_m \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ is endowed with the subdirect product satisfying condition (2).

Theorem 12. *The full list of normal subgroups of $W = S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:*

1. ***Type** T_{023} contains $E \wr\wr \tilde{A}_n \wr\wr H$, $\widetilde{T_n} \wr H$, where $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. There are 6 subgroups.*
2. ***The second type of subgroups is a subclass in** T_{023} with a new base of wreath product subgroup $\tilde{A}_{n^2}$: $E \wr S_n \wr\wr \tilde{A}_{n^2}$, $E \wr N_i(S_n \wr S_n)$. Therefore, this class has 12 new subgroups. Thus, the total number of normal subgroups in **Type** T_{023} is 18.*
3. ***Type** T_{003}: $A^{(3)}_{00(n^2)}$, $\widetilde{T}_{n^2}$, $\widetilde{T_n}^{(3)}$.*
4. ***Type** T_{123}: $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ and $N_i(S_n \wr S_n) \wr\wr \tilde{A}_{n^2}$. Thus, there are 29 new normal subgroups in T_{123}, taking into account repetition.*

Remark 13. Note that $E \wr \widetilde{A}_n^{(1)} \simeq E \wr (E \wr\wr \widetilde{A}_n)$ is contained in the family $E \wr N_i(S_n \wr S_n)$.

The condition of invariance of subgroups of H with depth l [8] of a permutation wreath product W is presented in the following theorem.

Definition 14. The set of elements from $\overset{k}{\underset{i=0}{\wr}} S_{n_i}$, $n_i \geqslant 3$, with depth k satisfying the following condition

$$\sum_{i=(s-1)n^m+1}^{sn^m} \operatorname{dec}([a_i]_k) = 2t, \quad t \in \mathbb{N},\ 1 \leq s \leq n^{k-m},\ [a_i]_j = e \text{ for } j = \overline{0, k-1}, \tag{4}$$

is called $\widetilde{A}^{(k)}_{n^m}$, where $1 \leq m < k$.

Theorem 15. *If $H \lhd W$, where $d(H) = l$ and $l : k - l = m$, then $A^{(k)}_{n^m} \lhd H$.*

We denote by $\mathrm{Aut}_f\, X^*$ the group of all finite automorphisms of a spherically homogeneous rooted tree.

Theorem 16. *Let $H \lhd \mathrm{Aut}_f\, X^*$ have depth k; then H contains a k-th level subgroup P having all even vertex permutations $p_{ki} \in A_n$ on X^k and trivial permutations in vertices of the rest of the levels.*

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