On some properties of affinor-semisymmetric spaces of class one

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Consider a pseudo-Riemannian space V_n of class one with a metric tensor $g_{ij}(x)$, in which an affinor $F_i^h(x)$ is defined. The tensor characteristics of class one spaces take the form

$$R_{hijk} = e(b_{hk}b_{ij} - b_{hj}b_{ik}), \quad b_{ij,k} - b_{ik,j} = 0, \tag{1}$$

where b_{ij} is the second fundamental form of the space V_n, and $e = \pm 1$. For spaces of class one, the following holds:

$$R_{ij} = e(bb_{ij} - b_{i\alpha}b_j^\alpha), \quad b \overset{def}{=} b_{\alpha\beta}g^{\alpha\beta} \tag{2}$$

Suppose that the affinor structure defined in this space is antisymmetric, semisymmetric, and almost complex, i.e., such that it satisfies

$$F_{ij,[kl]} = 0, \quad F^h_{\cdot\alpha}F^\alpha_{\cdot i} = -\delta_i^h, \quad F_{(ij)} = 0. \tag{3}$$

Spaces for which the first condition holds will be called affinor-semisymmetric. In spaces with an affinor structure, the operation of conjugation is defined. For such spaces, from the integrability conditions of the affinor, taking into account the Ricci identity, it follows that

$$R_{ij\overline{kl}} = R_{ijkl}. \tag{4}$$

Multiply the identity (1) by b_l^h, sum over the index h, perform the index substitution $l \to h$, and conjugate with respect to the indices j, k

$$b_{h\alpha}R^\alpha_{\cdot i\bar{j}\bar{k}} = bR_{hi\bar{j}\bar{k}} + R_{h\bar{j}}b_{i\bar{k}} - R_{h\bar{k}}b_{i\bar{j}}, \tag{5}$$

or, taking into account (4)

$$R_{hj}b_{ik} - R_{hk}b_{ij} - R_{h\bar{j}}b_{i\bar{k}} + R_{h\bar{k}}b_{i\bar{j}} = 0. \tag{6}$$

Contracting this expression over the indices i, k, we obtain

$$R_{hj} = \frac{R}{2b}(b_{hj} + b_{\bar{h}\bar{j}}). \tag{7}$$

The integrability conditions of an curvature tensor, taking into account the Ricci identity, take the form

$$R_{hijk,[\bar{l}\bar{p}]} = R_{hijk,[lp]}. \tag{8}$$

Therefore, substituting (1) into this expression and contracting first with respect to h, l, and then with respect to p, k, we obtain $RR_{ij} = 0$. From this, we obtain that either the Ricci tensor $R_{ij} = 0$ or the scalar curvature $R = 0$, but zero scalar curvature is a consequence of the Ricci tensor being zero. Moreover, the converse also holds. Indeed, if $R = 0$, substituting this into expression (7) yields $R_{ij} = 0$. Thus, we have proved:

Theorem 1. *In affinor-semisymmetric spaces of class one, the Ricci tensor is equal to zero.*

References

[1] Luther P. Eisenhart. *Riemannian Geometry.* Princeton: Princeton University Press, 1997.