Local behavior of ring Q-homeomorphisms with respect to the p-modulus under exponential integrability of the function Q

Ruslan Salimov
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: `ruslan.salimov1@gmail.com`

Mariia Stefanchuk
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: `stefanmv43@gmail.com`

Let a family of paths Γ be given in the complex plane $\mathbb{C}$. A Borel function $\rho : \mathbb{C} \to [0, \infty]$ is called *admissible* for Γ, denoted by $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)|dz| \geqslant 1$$

for every (locally rectifiable) path $\gamma \in \Gamma$.

Let $p > 1$. Recall that the *p-modulus* of the family of paths Γ is defined by

$$\mathrm{M}_p(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^p(z)\,dxdy.$$

For sets E_1, E_2, and E in $\mathbb{C}$, denote by $\Delta(E_1, E_2; E)$ the family of all paths $\gamma : [a, b] \to \mathbb{C}$ that join E_1 and E_2 in E, that is, $\gamma(a) \in E_1$, $\gamma(b) \in E_2$, and $\gamma(t) \in E$ for all $t \in (a, b)$.

Let G be a domain in the complex plane $\mathbb{C}$, let $z_0 \in G$, and let $Q : G \to [0, \infty]$ be a measurable function. A homeomorphism $f : G \to \mathbb{C}$ is called a *ring Q-homeomorphism with respect to the p-modulus at the point z_0* if

$$\mathrm{M}_p(\Delta(fC_1, fC_2; f\mathbb{A})) \;\leqslant \int\limits_{\mathbb{A}} Q(z) \cdot \eta^p(|z - z_0|)\,dxdy$$

for every ring $\mathbb{A} = \mathbb{A}(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < \mathrm{dist}(z_0, \partial G)$, for the circles $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$, $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\,dr = 1$.

Further, we use the following notation:

$$B_r = \{z \in \mathbb{C} : |z| \leqslant r\}, \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}.$$

In the following theorem, an upper estimate for the area of the image of a disk under ring Q-homeomorphisms with respect to the p-modulus is obtained.

Theorem 1. *Let $f : \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to the p-modulus at the point $z_0 = 0$ and let $1 < p < 2$. Suppose that for some numbers $\lambda > 0$ and $r_0 \in (0, 1)$ the following condition is fulfilled:*

$$Q_0 = \int\limits_{B_{r_0}} \exp\left(\lambda Q(z)\right) dxdy < \infty.$$

Then for all $r \in (0, r_0/2)$ the estimate

$$|fB_r| \leqslant c_0\,|B_r| \left(\log \frac{Q_0}{4|B_r|}\right)^{\frac{2}{2-p}},$$

holds, where $c_0 = \left(\frac{2}{\lambda}\right)^{\frac{2}{2-p}} \left(\frac{p-1}{2-p}\right)^{\frac{2(p-1)}{2-p}}$.

Below we present a theorem on the power-logarithmic asymptotics in terms of lower limits for ring Q-homeomorphisms with respect to the p-modulus at the point $z_0 = 0$ for $1 < p < 2$.

Theorem 2. *Let $f : \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to the p-modulus at the point $z_0 = 0$, let $1 < p < 2$, and let $f(0) = 0$. Assume that for some numbers $\lambda > 0$ and $r_0 \in (0,1)$ the condition*

$$Q_0 = \int\limits_{B_{r_0}} \exp\left(\lambda Q(z)\right) dxdy < \infty$$

holds. Then

$$\liminf_{z\to 0} \frac{|f(z)|}{|z|\left(\log\frac{1}{|z|}\right)^{\frac{1}{2-p}}} \leqslant k_0,$$

where $k_0 = \left(\frac{4}{\lambda}\right)^{\frac{1}{2-p}} \left(\frac{p-1}{2-p}\right)^{\frac{p-1}{2-p}}$.

Research supported by the National Research Foundation of Ukraine, Project number 2025.07/0014, Project name: "Modern problems of Mathematical Analysis and Geometric Function Theory".