A gauge theoretical generalisation of Bryant's correspondence

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A classical theorem in the theory of minimal surfaces establishes a correspondence between minimal surfaces in $\mathbb{R}^n$ and null holomorphic curves in $\mathbb{C}^n$. A hyperbolic version of this correspondence is due to Bryant: null holomorphic curves in $\mathrm{SL}(2,\mathbb{C})$ correspond to CMC-1 surfaces in the hyperbolic space $\mathbb{H}^3$. We also have a relativistic Bryant type correspondence: CMC-1 immersions in the hyperbolic space are replaced by space-like CMC-1 immersions in the de Sitter space.

We prove a mutual generalisation of all these results: let H be a real Lie group, $\pi : P \to M$ a principal H-bundle, A a connection on P and $\alpha \in A^1_{\mathrm{Ad}}(P,\mathfrak{h})$ a tensorial 1-form of type Ad which induces isomorphisms $A_\xi \to \mathfrak{h}$.

Such a pair (α, A) defines an almost complex structure J^α_A on P, which is integrable if and only if (α, A) solves a gauge-invariant non-linear first order differential system. A non-degenerate symmetric Ad_H-invariant bilinear form g on $\mathfrak{h}$ defines pseudo-Riemannian metrics g^α_M, $\mathfrak{g}^\alpha_A$ on M, respectively P, and a non-degenerate bilinear form $\omega^{\alpha,g}_A : T_P \times_P T_P \to \mathbb{C}$ which is holomorphic when J^α_A is integrable.

Assuming that this is the case, we have a Bryant type correspondence between space-like, $\omega^{\alpha,g}_A$-isotropic holomorphic immersions $Y \to P$ and space-like conformal immersions $Y \to (M, g^\alpha_M)$ whose mean curvature vector field is given by a simple explicit formula.

In particular, one obtains such a correspondence for any principal bundle of the form $G \to G/H$, where G is a complex Lie group, and H is a real form of G endowed with a non-degenerate, Ad_H-invariant, symmetric bilinear form g on its Lie algebra $\mathfrak{h}$.

References