Menshov–Trokhymchuk type theorem in a finite-dimensional Banach algebra

Maksym Tkachuk
Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv, Ukraine
E-mail: maxim.v.tkachuk@gmail.com

We weaken the conditions of monogenity for functions that take values in an arbitrary finite-dimensional commutative algebra over the field of complex numbers. The monogenity of the function is understood as a combination of its continuity and the existence of the Gâteaux derivative [3].

The main result generalizes the classical Menchov theorem, its part related to the K''' property, to the case of functions monogenic in finite-dimensional algebras, when functions are defined in a specific linear span of the algebra [1].

Consider an n-dimensional commutative associative algebra $\mathbb{A}_n^m$ with unit 1 over the field $\mathbb{C}$, where there exists a Cartan's basis $\{I_k\}_{k=1}^n$ and norm

$$\|\lambda_1 I_1 + \lambda_2 I_2 + \ldots + \lambda_n I_n\| := \sqrt{|\lambda_1|^2 + |\lambda_2|^2 + \ldots + |\lambda_n|^2}.$$

The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{r=1,\, r\neq u}^{n} \lambda_r I_r : \lambda_r \in \mathbb{C} \right\}, \quad u = 1, 2, \ldots, m.$$

Consider m linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ satisfying the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall\, \omega \in \mathcal{I}_u, \quad u = 1, 2, \ldots, m.$$

Fix a real k-dimensional subspace

$$E_k := \{\zeta = x_1 e_1 + x_2 e_2 + \ldots + x_k e_k \,:\, x_1, \ldots, x_k \in \mathbb{R}\} \subset \mathbb{A}_n^m,$$

where the vectors $e_1, \ldots, e_k$ are linearly independent over the field $\mathbb{R}$. We will impose the following restriction on the choice of the linear span E_k:

$$\{f_u(\zeta) : \zeta \in E_k\} = \mathbb{C}, \qquad u = 1, 2, \ldots, m, \tag{1}$$

i.e., the images of the set E_k under all mappings f_u must be the whole complex plane.

The intersection of some maximal ideal $\mathcal{I}_u$, $u \in [1, m] \cap \mathbb{N}$, of the algebra $\mathbb{A}_n^m$ with the linear space E_k is a certain plane L_u of dimension $k-2$. The preimage in E_k of an arbitrary point $\xi \in \mathbb{C}$ under the mapping f_u is the plane

$$L_u^\zeta := \{\zeta + \tau : \tau \in L_u\},$$

where ζ is some element of E_k such that $\xi = f_u(\zeta)$. Obviously, the plane L_u^ζ is parallel to the plane L_u.

Definition 1. We say that a function $\Phi : \Omega \to \mathbb{A}_n^m$ satisfies the condition $K'''_{\mathbb{A}_n^m, E_k}$ at the point $\zeta \in \Omega \subset E_k$, if there exists an element $\Phi_*(\zeta) \in \mathbb{A}_n^m$ such that the equality

$$\lim_{\delta \to 0+0} \left(\Phi(\zeta + \delta h) - \Phi(\zeta)\right) \delta^{-1} = h \Phi_*(\zeta) \tag{2}$$

holds for k distinct vectors $h_1, h_2, \ldots, h_k$ that form a basis in the space E_k, and also such that for each mapping f_u, $u \in [1, m] \cap \mathbb{N}$, there are exactly two of them whose images under the mapping f_u are noncollinear in $\mathbb{C}$.

Theorem 2. *Let a domain $\Omega \subset E_k$ have the property that its intersections with the planes L_u^ζ are connected, where $\zeta \in \Omega$, $u = \overline{1, m}$, and let a function $\Phi : \Omega \to \mathbb{A}_n^m$, continuous in Ω, satisfy the condition $K'''_{\mathbb{A}_n^m, E_k}$ at all points $\zeta \in \Omega$, except for at most a countable set of points. Then the function Φ is monogenic in the domain Ω.*

References

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