Symmetry properties of symmetric Nyzhnyk models

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Applying the inverse scattering method to integrating multidimensional nonlinear equations in [2], Leonid Nyzhnyk constructed one of the first multidimensional integrable models

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y, \tag{1}$$

which is now called the *Nyzhnyk system*. Since k_1 and k_2 are arbitrary constant parameters subject to the condition $(k_1, k_2) \neq (0, 0)$, the system (1) in fact represents a class of systems, which is the union of two cosets with respect to scaling equivalence transformations and the permutation of (x, v^1) and (y, v^2). Namely, in the *symmetric* case, where both parameters k_1 and k_2 are nonzero, $k_1 k_2 \neq 0$, one can set $(k_1, k_2) = (1, 1)$, whereas in the *asymmetric* case, where exactly one of them vanishes, one can set $(k_1, k_2) = (1, 0)$. Using the introduction of potentials, limiting procedures, interpretations of independent and/or dependent variables as real or complex, and differential substitutions, from the system (1) one can derive a considerable number of various integrable models, which we refer to as *Nyzhnyk models* [5].

Among these models, one distinguishes symmetric and asymmetric, dispersive and dispersionless, standard models with quadratic nonlinearities and modified models with cubic nonlinearities, single partial differential equations and systems of such equations, real, complex, mixed, and specific models, where complex variables are split into pairs of complex-conjugate variables, as well as linear and nonlinear nonisospectral Lax representations in the dispersive and dispersionless cases, respectively. According to the sign of nonlinearities, modified models are additionally divided into defocusing and focusing ones. There also exist generalizations of the Nyzhnyk models with larger numbers of independent or dependent variables. The classification of Nyzhnyk models presented in [5] makes it possible to clearly identify structural and functional distinctions between groups of these models, thereby creating a basis for their systematic analysis as well as for revealing their genetic relations with other equations of mathematical physics.

Each of the Nyzhnyk models possesses interesting symmetry properties but they are not well studied. Thus, the extended classical symmetry analysis has been carried out completely only for the (real symmetric potential) dispersionless Nyzhnyk equation [1, 3, 4]. At the same time, relations between the maximal Lie invariance pseudoalgebras of Nyzhnyk models according to various relations between these models have been revealed. In particular, it turned out that the maximal Lie invariance pseudoalgebra of each dispersive Nyzhnyk model is a codimension-one pseudosubalgebra of the maximal Lie invariance pseudoalgebra of its dispersionless counterpart, where the complementary part is associated with scaling transformations of the spatial variables. Moreover, in order to derive the maximal Lie invariance pseudoalgebra of a Lax representation of a Nyzhnyk model, it suffices to prolong the elements of the pseudoalgebra of this model to the pseudopotential regarded as a new dependent variable and to add one more generating vector field associated with the arbitrariness in the definition of the pseudopotential.

There exists a relation between the point-symmetry pseudogroups of the corresponding dispersionless and dispersive Nyzhnyk models that is similar to that between their maximal Lie invariance pseudoalgebras. As an example, consider the (real symmetric potential) dispersive and dispersionless Nyzhnyk equations

$$u_{txy} = u_{xxxxy} + u_{xyyyy} + (u_{xx} u_{xy})_x + (u_{yy} u_{xy})_y, \tag{2}$$
$$u_{txy} = (u_{xx} u_{xy})_x + (u_{yy} u_{xy})_y. \tag{3}$$

The maximal Lie invariance pseudoalgebras of the equations (2) and (3) are the (infinite-dimensional) pseudosubalgebras

$$\mathfrak{g}_2 := \langle D^t(\tau),\, P^x(\chi),\, P^y(\rho),\, R^x(\alpha),\, R^y(\beta),\, Z(\sigma)\rangle, \quad \mathfrak{g}_3 := \mathfrak{g}_2 + \langle D^{\rm s}\rangle,$$

respectively, where

$$D^t(\tau) = \tau\partial_t + \tfrac{1}{3}\tau_t x\partial_x + \tfrac{1}{3}\tau_t y\partial_y - \tfrac{1}{18}\tau_{tt}(x^3+y^3)\partial_u, \quad D^{\rm s} = x\partial_x + y\partial_y + 3u\partial_u,$$
$$P^x(\chi) = \chi\partial_x - \tfrac{1}{2}\chi_t x^2\partial_u, \quad P^y(\rho) = \rho\partial_y - \tfrac{1}{2}\rho_t y^2\partial_u,$$
$$R^x(\alpha) = \alpha x\partial_u, \quad R^y(\beta) = \beta y\partial_u, \quad Z(\sigma) = \sigma\partial_u.$$

Thus, the pseudoalgebra $\mathfrak{g}_2$ is a codimension-one pseudosubalgebra of the pseudoalgebra $\mathfrak{g}_3$.

Theorem 1. (i) *The point-symmetry pseudogroup G_3 of the dispersionless Nyzhnyk equation (3) is generated by the transformation $\mathscr{J}$: $\tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde u = u$ and the transformations of the form*

$$\begin{aligned}&\tilde t = T(t), \quad \tilde x = CT_t^{1/3}x + X^0(t), \quad \tilde y = CT_t^{1/3}y + Y^0(t),\\ &\tilde u = C^3u - \frac{C^3T_{tt}}{18T_t}(x^3+y^3) - \frac{C^2}{2T_t^{1/3}}(X^0_t x^2 + Y^0_t y^2) + W^1(t)x + W^2(t)y + W^0(t),\end{aligned} \tag{4}$$

where T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t \neq 0$, and C is an arbitrary nonzero constant.

(ii) *The contact-symmetry pseudogroup $G_3^{\rm c}$ of the dispersionless Nyzhnyk equation (3) coincides with the first prolongation of the pseudogroup G_3.*

(iii) *The point- and contact-symmetry pseudogroups G_2 and $G_2^{\rm c}$ of the dispersive Nyzhnyk equation (2) is a pseudosubgroup of the pseudogroups G_3 and $G_3^{\rm c}$ that are singled out by the constraint $C = 1$.*

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