Mixed volume inequalities for zonoids and log-submodularity of volume

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1. Zonoids and log-submodularity. A zonotope is the Minkowski sum of line segments—a linear projection of a cube. A zonoid is a Hausdorff limit of zonotopes. For any zonoid A and zonoids $B_1, \dots, B_k$ in $\mathbb{R}^n$, Fradelizi, Madiman, Meyer and Zvavitch conjectured the log-submodularity inequality

$$\mathrm{vol}(A) \cdot \mathrm{vol}(A + B_1 + \cdots + B_k) \;\le\; \prod_{i=1}^{k} \mathrm{vol}(A + B_i),$$

which expresses a multiplicative diminishing-returns principle for Minkowski addition. This property is distinct from the classical log-concavity of volume, which concerns Minkowski combinations rather than sums. The authors have proven the conjecture in dimension three in [2].

The *mixed volume* $V(K_1, \dots, K_n)$ is defined by the polarization identity

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n+j} \sum_{1 \le i_1 < \cdots < i_j \le n} \mathrm{vol}(K_{i_1} + \cdots + K_{i_j}),$$

and satisfies $V(K, \dots, K) = \mathrm{vol}(K)$.

Using mixed volumes, the conjecture is equivalent to a family of inequalities that we call *Bezout-type inequalities*:

$$\mathrm{vol}(A)^{k-1}\, V(A[n-k], B_1, \dots, B_k) \;\le\; \frac{n^k (n-k)!}{n!} \prod_{i=1}^{k} V(A[n-1], B_i), \qquad 1 \le k \le n.$$

The name reflects a structural analogy with Bézout's theorem in algebraic geometry: just as the intersection number of hypersurfaces is bounded by the product of their degrees, here the mixed volume (measuring "intersection" of Minkowski sums) is bounded by a product of lower-order mixed volumes. This connection suggests a deep interplay between convex geometry and intersection theory.

2. Reduction to a hypercube inequality. For arbitrary dimension n, the Bezout-type inequality reduces to a purely geometric statement about the $(n-1)$-dimensional cube $[0,1]^{n-1}$. Assign non-negative weights s_p to the 2^{n-1} vertices. The inequality becomes

$$\Big(\sum_{p \in \{0,1\}^{n-1}} s_p \Big)^{n-2} \Big(\sum_{\substack{S \subset \{0,1\}^{n-1} \\ |S| = n}} \big(\prod_{p \in S} s_p\big) c_S \Big) \;\le\; \prod_{\text{facets } F \text{ of } [0,1]^{n-1}} \Big(\sum_{p \in F \cap \{0,1\}^{n-1}} s_p \Big),$$

where c_S denotes the normalized $(n-1)$-dimensional volume of the simplex with vertex set S. Thus the hypercube inequality compares a weighted total volume of all $(n-1)$-dimensional simplices inscribed in the cube with the product of the total weights on each facet.

3. Main theorem.

Theorem 1 (Hypercube inequality for $n = 4$). *Let $s_1, \ldots, s_8 \geq 0$ be vertex weights. Then*

$$\Big(\sum_{i=1}^{8} s_i\Big)^2 \Bigg[\Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{5\leq i<j<k\leq 8} s_i s_j s_k\Big) + \Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{1\leq i<j<k\leq 4} s_i s_j s_k\Big)$$
$$+ (s_1+s_2)(s_3+s_4)(s_5s_6+s_7s_8) + (s_1+s_3)(s_2+s_4)(s_5s_7+s_6s_8)$$
$$+ (s_1+s_4)(s_2+s_3)(s_6s_7+s_5s_8) + 2s_1s_2s_6s_7 + 2s_2s_3s_5s_8\Bigg]$$
$$\leq \Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{i\in\{1,3,5,7\}} s_i\Big)\Big(\sum_{i\in\{2,4,6,8\}} s_i\Big)\Big(\sum_{i\in\{1,2,5,6\}} s_i\Big)\Big(\sum_{i\in\{3,4,7,8\}} s_i\Big).$$

Equality holds iff at least one of the six face sums is zero.

4. Geometric consequences. The theorem directly implies the Bezout-type inequality for $n = 4$, hence the log-submodularity conjecture for zonoids in $\mathbb{R}^4$. Moreover, it confirms the projection inequality

$$\mathrm{vol}(A)^{k-1}\,\mathrm{vol}(P_{[b_1,\ldots,b_k]^\perp}A) \;\leq\; \prod_{i=1}^{k} \mathrm{vol}(P_{b_i^\perp}A)$$

for any orthonormal basis $\{b_1, \ldots, b_4\}$ and any zonoid $A \subset \mathbb{R}^4$ [1]. Geometrically, this bounds the volume of a zonoid projected onto a k-codimensional subspace by the product of its projections onto k coordinate hyperplanes. The equality case occurs when at least one of those projections has zero volume—a natural geometric degeneracy.

The algebraic certificate derived from the cube geometry suggests a systematic method for higher dimensions. The hypercube polynomial is invariant under the cube's symmetry group, and its Newton polytope has a rich combinatorial structure. The analogy with Bézout's theorem opens a promising link to intersection theory and toric geometry, which may guide the search for certificates in higher dimensions.

References

[1] Gennadiy Averkov, Katherina von Dichter, and Ivan Soprunov. Mixed volume inequalities for zonoids. Preprint, 2026.

[2] Matthieu Fradelizi, Mokshay Madiman, Mathieu Meyer, and Artem Zvavitch. On the volume of the Minkowski sum of zonoids. *J. Funct. Anal.*, 286(3):Paper No. 110247, 2024. doi:10.1016/j.jfa.2023.110247.