A criterion for finitely many intervals in achievement sets of random power series with restricted alphabet

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Let ξ_i be independent identically distributed random variables with values in the alphabet $A = \{a_0 = 0, a_1, \ldots, a_p = r \mid a_i \in \mathbb{N} \cup \{0\}, a_i > a_{i-1}\}$. We study the set of values of the random variable

$$\xi = \sum_{k=1}^{\infty} \xi_k s^{-k},$$

where $s \geq 2$ is a fixed natural number. We denote this set by $\mathcal{E}(\xi)$.

Definition 1. By a *cylindrical interval* of rank n of the half-open interval $[0,1)$ we mean the half-open interval

$$\widetilde{\Delta}_{\alpha_1\alpha_2\ldots\alpha_n} := \left[\sum_{k=1}^{n} \frac{\alpha_k}{s^k}, \sum_{k=1}^{n} \frac{\alpha_k}{s^k} + \frac{1}{s^n}\right),$$

where $\alpha_k \in \{0, 1, \ldots, s-1\}$.

Remark 2. Every cylindrical interval contains s pairwise disjoint cylindrical intervals of the next rank.

For a cylindrical interval we introduce the notation

$$t_j := \frac{\alpha_1}{s} + \ldots + \frac{\alpha_n}{s^n} - \frac{M-1-j}{s^n}, \quad j \in \{0, 1, \ldots, M-1\},$$

where $M = [\frac{r}{s}] + 1$. Thus t_{M-1} is the left endpoint of the cylindrical interval, and the points t_j are obtained by shifting this point to the left by $\frac{M-1-j}{s^n}$.

Definition 3. The *prefix of a cylindrical interval* of rank n is the ordered tuple

$$(p_0, p_1, \ldots, p_{M-1})$$

of zeros and ones, where $p_i = 1$ if

$$t_i \in \mathcal{E}(\sum_{k=1}^{n} \xi_k s^{-k}),$$

and $p_i = 0$ otherwise.

Definition 4. The *prefix graph* G' is the graph defined as follows:

1) its vertices are all possible prefixes of cylindrical intervals that can be reached from the vertices of the form

$$(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1);$$

2) for every vertex, s directed edges are defined, going from it to those vertices that are prefixes of the cylindrical intervals of the next rank contained in the given cylindrical interval.

Definition 5. A partial equivalence class is called *terminal* if no vertex of the class has a directed outgoing edge leading to a vertex of another class or to a transient vertex.

Theorem 6. *The set $\mathcal{E}(\xi)$ of values of a random power series contains an interval if and only if the corresponding directed prefix graph has a strongly connected component different from $\{(0, 0, \ldots, 0)\}$ from which there is no outgoing edge to another strongly connected component of the graph.*

Definition 7. The *colors of partial equivalence classes and transient vertices* are defined by the following rules:

The terminal class $\{(0, 0, \ldots, 0)\}$ is assigned the *white* color, while all other terminal classes are assigned the *black* color. Then, iteratively, each class (or transient vertex) is assigned a color depending on the colors of its immediate successors, namely:

- *white*, if all successors are white;
- *black*, if all successors are black;

- *black-and-white*, if there are successors of both colors, or if there is at least one black-and-white successor.

Lemma 8. *If the condensed graph $\bar{G}'$ contains a non-cyclic black-and-white class, then the set $\mathcal{E}(\xi)$ contains infinitely many intervals.*

Lemma 9. *If for two distinct black-and-white classes there exists a directed path from one of them to the other, then the set $\mathcal{E}(\xi)$ contains infinitely many intervals.*

Let an edge $v \rightarrowtail w$ of the graph G' correspond to the transition from a cylindrical interval $\widetilde{\Delta}_{\alpha_1\ldots\alpha_n}$ to its subcylinder $\widetilde{\Delta}_{\alpha_1\ldots\alpha_n d}$, $(d \in \{0, 1, \ldots, s-1\})$. The number d will be called the *index* of the edge and denoted by $\operatorname{ind}(v, w)$.

Let v be a vertex of some cyclic partial equivalence class C, and let k be the index of the unique edge that goes within the class to a vertex $v_2 \in C$. For v, we split the set of its other immediate successors into two sets

$$S^-(v) := \{\, w \mid v \rightarrowtail w,\ \operatorname{ind}(v, w) < k\},$$
$$S^+(v) := \{\, w \mid v \rightarrowtail w,\ \operatorname{ind}(v, w) > k\}.$$

Definition 10. We say that a vertex v of a cyclic class has *property* D_1 if all vertices in $S^-(v)$ are black and all vertices in $S^+(v)$ are white. Analogously, the vertex v has *property* D_2 if all vertices in $S^-(v)$ are white and all vertices in $S^+(v)$ are black.

Lemma 11. *For the set $\mathcal{E}(\xi)$ to contain only finitely many intervals it is necessary that, in every black-and-white partial equivalence class, all vertices have the same property — either all have D_1, or all have D_2.*

Theorem 12. Criterion for the finiteness of the number of intervals in $\mathcal{E}(\xi)$. *The set $\mathcal{E}(\xi)$ contains only finitely many intervals if and only if the following conditions are simultaneously satisfied:*

- *1) all black-and-white classes are cyclic;*
- *2) there are no directed paths between distinct black-and-white classes;*
- *3) for every black-and-white class, all of its vertices have the same property — either D_1 or D_2.*

References

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