Fractional harmonic measure as a tool in minimum Riesz energy problems with external fields

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For the Riesz kernel $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ on $\mathbb{R}^n$, where $n \geqslant 2$, $\alpha \in (0,2]$, and $\alpha < n$, we consider the problem of minimizing the Gauss functional

$$\int \kappa_\alpha(x,y)\,d(\mu\otimes\mu)(x,y) + 2\int f_{q,z}\,d\mu, \quad \text{where } f_{q,z} := -q\int \kappa_\alpha(\cdot,y)\,d\varepsilon_z(y),$$

q being a positive number, ε_z the unit Dirac measure at a fixed point $z \in \mathbb{R}^n$, and μ ranging over all probability measures of finite energy, concentrated on a *quasiclosed* set $A \subset \mathbb{R}^n$ (i.e. such A that can be approximated in outer Riesz capacity $c^*(\cdot)$ by closed sets, cf. [2]). For any

$$z \in A^u \cup (\mathbb{R}^n \setminus \mathrm{Cl}_{\mathbb{R}^n} A),$$

where A^u is the set of all inner α-ultrairregular points for A, we provide necessary and sufficient conditions for the existence of the minimizer $\lambda_{A,f_{q,z}}$, establish its alternative characterizations, and describe its support, thereby discovering new interesting phenomena. In detail, $z \in \partial_{\mathbb{R}^n} A$ is said to be *inner α-ultrairregular* for A if the inner α-harmonic measure ε_z^A [3] is of finite energy, or alternatively if the inverse A_z^* of A with respect to the unit sphere centered at z is of finite inner capacity $c_*(A_z^*)$. We show that for any

$$z \in A^u \cup (\mathbb{R}^n \setminus \mathrm{Cl}_{\mathbb{R}^n} A), \qquad \lambda_{A,f_{q,z}}$$

exists if and only if either $c_*(A) < \infty$, or $q \geqslant H_z$, where

$$H_z := 1/\varepsilon_z^A(\mathbb{R}^n) \in [1,\infty).$$

Thus, for any closed A, any $z \in A^u$, and any $q \geqslant H_z$ — even arbitrarily large, no compensation effect occurs between the two oppositely signed charges, $-q\varepsilon_z$ and $\lambda_{A,f_{q,z}}$, carried by the same conductor A, which at first glance seems to contradict our physical intuition. Another interesting phenomenon is that, if A is closed and of unbounded boundary, then for any $z \in A^u \cup (\mathbb{R}^n \setminus A)$, the support of $\lambda_{A,f_{H_z,z}}$ is noncompact, whereas that of $\lambda_{A,f_{H_z+\beta,z}}$ is already compact for any $\beta \in (0,\infty)$ — even arbitrarily small. The above-mentioned results were published in the author's recent paper [5], and they substantially improve some of the latest ones on the problem in question, e.g. those in [1, 4].

References

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