Sturm-Liouville problems with a boundary condition depending linearly on an eigenparameter

Yagub Aliyev, Narmin Aliyeva

ADA University, Baku State University

Sturm-Liouville problems with an eigenparameter in a boundary condition

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{1.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta,\ 0 \le \beta < \pi, \tag{1.2}$$

$$(a\lambda + b)y(1) = (c\lambda + d)y'(1), \tag{1.3}$$

where a, b, c, d are real constants, $ad - bc < 0$, λ is the spectral parameter, and $q(x)$ is a real valued and continuous function over the interval $[0, 1]$.

History of the problem

The case $ad - bc > 0$ was studied in

Charles T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **77(3-4)** (1977), 293-308.

N. B. Kerimov and V. S. Mirzoev, *On the basis properties of one spectral problem with a spectral parameter in boundary conditions*, Siberian Math. J. **44** (2003), 813-816.

History of the problem

The case $ad - bc < 0$ was also studied in the following paper and their approach used the introduction of the exit space $L_2 \oplus \mathbb{C}$ with an indefinite metric.

Z.S. Aliyev, E.A. Aghayev, The basis properties of the system of root functions of Sturm-Liouville problem with spectral parameter in the boundary condition, Proceedings of IMM of NAS of Azerbaijan. **32** (2010), 63-72.

Sturm-Liouville problems with an eigenparameter in the boundary conditions

This method and the solution of more general problem about the basis properties were discussed in

E.M. Russakovskii, Operator treatment of a boundary-value problem with the spectral parameter appearing rationally in the boundary conditions (in Russian). Theory Funct. Funct. Anal. Appl. 30, (1978) 120-128.

A. A. Shkalikov, *Boundary value problems for ordinary differential equations with a parameter in the boundary conditions* (Russian. English summary), Tr. Semin. Im. I. G. Petrovskogo **9** (1983) 190-229.

Special case of the boundary condition

The special problem similar to (1.1)-(1.3) but with the condition (1.2) replaced by more special $y'(0) = 0$ and other related problems, were studied in

A. A. Shkalikov, *Basis Properties of Root Functions of Differential Operators with Spectral Parameter in the Boundary Conditions*, Diff. Equat., **55:5** (2019), 631–643.

The current study does not use the exit space $L_2 \oplus \mathbb{C}$ and its indefinite metric.

Eigenvalues

It was shown in the following paper that the eigenvalues of the boundary value problem (1.1)–(1.3) form an infinite sequence with the only accumulation point at $+\infty$. Moreover, exactly one of the following alternatives occurs:

(i) all eigenvalues are real and simple;

(ii) all eigenvalues are real and, except for a single eigenvalue of algebraic multiplicity 2, are simple;

(iii) all eigenvalues are real and, except for a single eigenvalue of algebraic multiplicity 3, are simple;

(iv) all eigenvalues are simple, and apart from one conjugate pair of non-real eigenvalues, they are all real.

P. A. Binding and P. J. Browne, *Application of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions*, Proc. Roy. Soc. Edinburgh **125A** (1995), 1205-1218.

Asymptotics

The eigenvalues λ_n ($n \geq 0$) will be considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. The asymptotics of the eigenvalues is the following formula

$$\lambda_n = \begin{cases} (n-1)^2 \pi^2 + O(1), & \text{if } \beta \neq 0, \\ \left(n - \frac{1}{2}\right)^2 \pi^2 + O(1), & \text{if } \beta = 0. \end{cases} \tag{1.4}$$

P. A. Binding, P. J. Browne and B. A. Watson, J. Math. Anal. Appl. 291 (2004), 246-261.

Asymptotics

Using (1.4) one can show that the root functions with one function removed form a system quadratically close to the trigonometric systems. This method of using the quadratic closeness and Bari's theorem for the completeness in $L_2(0,1)$ first appeared in the paper of A. G. Kostyuchenko and A. V. Skorokhod. The method of using the biorthogonal system for the minimality is due to E.I. Moiseev and N.Yu. Kapustin.

A. G. Kostyuchenko, A. V. Skorokhod, *On a theorem of N. K. Bari*, Uspekhi Mat. Nauk, 1953 Volume 8, Issue 5(57), Pages 165-166. (in Russian)

E. I. Moiseev and N. Yu. Kapustin, *On the singularities of the root space of a spectral problem with a spectral parameter in the boundary condition*, Dokl. Akad. Nauk, **66(1)** (2002), 14-18.

Motivation

We are interested in the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the root function system of the boundary value problem (1.1)-(1.3). The cases $a = 0$ and $c = 0$ were studied in the following papers, respectively.

Y. N. Aliyev, *Minimality of the system of root functions of Sturm-Liouville problems with decreasing affine boundary conditions*, Colloquium Mathematicum **109(1)** (2007), 147-162.

Y.N. Aliyev, Minimality Properties of Sturm-Liouville Problems with Increasing Affine Boundary Conditions. In: Bastos M.A., Castro L., Karlovich A.Y. (eds) Operator Theory, Functional Analysis and Applications. Operator Theory: Advances and Applications, Birkhäuser, Cham. vol 282 (2021) 33-49.

Notation

By (1.1)-(1.3), if y_n is an eigenfunction corresponding to λ_n, then

$$-y_n'' + q(x)y_n = \lambda_n y_n, \tag{2.1}$$

$$y_n'(0)\sin\beta = y_n(0)\cos\beta, \tag{2.2}$$

$$(a\lambda_n + b)y_n(1) = (c\lambda_n + d)y_n'(1). \tag{2.3}$$

We denote by $(\cdot, \cdot)$ the scalar product in $L_2(0,1)$, and by $\|\cdot\|_p$ the norm in $L_p(0,1)$.

Inner Products of eigenfunctions

Lemma

Let y_n, y_m be eigenfunctions corresponding to the eigenvalues λ_n and λ_m such that $\lambda_n \neq \overline{\lambda_m}$.
(a) If $\lambda_n, \lambda_m \neq -\frac{d}{c}$, then

$$(y_n, y_m) = -(ad - bc) \frac{y_n(1)\overline{y_m(1)}}{(c\lambda_n + d)\left(c\overline{\lambda}_m + d\right)}. \tag{2.4}$$

(b) If $\lambda_n = -\frac{d}{c} \neq \lambda_m$, then

$$(y_n, y_m) = -(ad - bc) \frac{y_n'(1)\overline{y_m(1)}}{(a\lambda_n + b)(c\overline{\lambda_m} + d)}. \tag{2.5}$$

Corollary

If λ_r is a non-real eigenvalue, then

$$\|y_r\|_2^2 = -(ad - bc)\frac{|y_r(1)|^2}{|c\lambda_r + d|^2}. \tag{2.9}$$

First associated function

If λ_k is a double eigenvalue ($\lambda_k = \lambda_{k+1}$) or if λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$), then for the associated function y_{k+1} corresponding to the eigenfunction y_k, the following relations hold:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k, \tag{2.10}$$

$$y_{k+1}(0)\cos\beta = y'_{k+1}(0)\sin\beta, \tag{2.11}$$

$$(a\lambda_k + b)\,y_{k+1}(1) + ay_k(1) = (c\lambda_k + d)\,y'_{k+1}(1) + cy'_k(1). \tag{2.12}$$

Note that $y_{k+1} + cy_k$, where c is an arbitrary constant, is also an associated function. So the associated function y_{k+1} is not unique.

M. A. Naimark, *Linear differential operators*, Parts I, II, Ungar, New York, 1967, 1968.

Lemma

Suppose that λ_k is a multiple eigenvalue $(\lambda_k = \lambda_{k+1} \leq \lambda_{k+2})$.
(a) If $\lambda_k \neq -\frac{d}{c}$, then

$$\|y_k\|_2^2 = -(ad - bc)\frac{y_k^2(1)}{(c\lambda_k + d)^2}. \tag{2.13}$$

(b) If $\lambda_k = -\frac{d}{c}$, then

$$\|y_k\|_2^2 = -(ad - bc)\frac{(y_k'(1))^2}{(a\lambda_k + b)^2} \tag{2.14}$$

Let

$$B_n = \begin{cases} \|y_n\|_2^2 + (ad-bc)\dfrac{|y_n(1)|^2}{|c\lambda_n+d|^2}, & \lambda_n \neq -\dfrac{d}{c}, \\ \|y_n\|_2^2 + (ad-bc)\dfrac{\left(y_n'(1)\right)^2}{(a\lambda_n+b)^2}, & \lambda_n = -\dfrac{d}{c}. \end{cases} \tag{2.16}$$

Lemma

$B_n \neq 0$ if and only if the corresponding eigenvalue λ_n is real and has multiplicity one (is simple).

Lemma

If λ_r and $\lambda_s = \overline{\lambda_r}$ ($s := r+1$) form a conjugate pair of non-real eigenvalues, then

$$T_s = \overline{T_r} := (y_s, y_r) + (ad - bc)\frac{y_s^2(1)}{(c\lambda_s + d)^2} \neq 0. \tag{2.20}$$

Inner products with the first associated function.

Lemma

Suppose that λ_k is an eigenvalue of multiplicity two or three $(\lambda_k = \lambda_{k+1} \leq \lambda_{k+2})$.
(a) If $\lambda_k \neq -\frac{d}{c} \neq \lambda_n$, then

$$(y_{k+1}, y_n) = -(ad - bc)\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{cy_k(1)}{(c\lambda_k + d)^2}\right) \cdot \frac{y_n(1)}{c\lambda_n + d}. \tag{3.1}$$

(b) If $\lambda_k \neq -\frac{d}{c} = \lambda_n$, then

$$(y_{k+1}, y_n) = -(ad - bc)\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{cy_k(1)}{(c\lambda_k + d)^2}\right) \cdot \frac{y_n'(1)}{a\lambda_n + b}. \tag{3.2}$$

(c) If $\lambda_k = -\frac{d}{c} \neq \lambda_n$, then

$$(y_{k+1}, y_n) = -(ad - bc)\left(\frac{y_{k+1}'(1)}{a\lambda_k + b} - \frac{ay_k'(1)}{(a\lambda_k + b)^2}\right) \cdot \frac{y_n(1)}{c\lambda_n + d}. \tag{3.3}$$

Lemma

Suppose that λ_k *is an eigenvalue of multiplicity two and not three* $(\lambda_k = \lambda_{k+1} < \lambda_{k+2})$.
(a) If $\lambda_k \neq -\frac{d}{c}$, *then*

$$T_k := (y_{k+1}, y_k) + (ad - bc)\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{c y_k(1)}{(c\lambda_k + d)^2}\right) \cdot \frac{y_k(1)}{c\lambda_k + d} \neq 0. \tag{3.6}$$

(b) If $\lambda_k = -\frac{d}{c}$, *then*

$$S_k := (y_{k+1}, y_k) + (ad - bc)\left(\frac{y'_{k+1}(1)}{a\lambda_k + b} - \frac{a y'_k(1)}{(a\lambda_k + b)^2}\right) \cdot \frac{y'_k(1)}{a\lambda_k + b} \neq 0. \tag{3.7}$$

Second associated function

If λ_k is an eigenvalue of multiplicity three, i.e., $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, then in addition to the first-order associated function y_{k+1} defined by (2.10)-(2.12), there exists a second-order associated function y_{k+2}:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}, \tag{4.1}$$

$$y_{k+2}(0)\cos\beta = y'_{k+2}(0)\sin\beta, \tag{4.2}$$

$$(a\lambda_k + b)\,y_{k+2}(1) + ay_{k+1}(1) = (c\lambda_k + d)\,y'_{k+2}(1) + cy'_{k+1}(1). \tag{4.3}$$

Similar to y_{k+1} the associated function y_{k+2} is not unique, because the function $y_{k+2} + dy_k$, where d is a constant, is also an associated function of the second order. Note also that the second order associated function $y_{k+2} + cy_{k+1}$ corresponds to the first order associated function $y_{k+1} + cy_k$.

M. A. Naimark, *Linear differential operators*, Parts I, II, Ungar, New York, 1967, 1968.

Lemma

Suppose that λ_k is an eigenvalue of multiplicity three $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$.

(a) If $\lambda_k \neq -\frac{d}{c} \neq \lambda_n$, then

$$(y_{k+2}, y_n) = -(ad - bc)\left(\frac{y_{k+2}(1)}{c\lambda_k + d} - \frac{c\, y_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right) \cdot \frac{y_n(1)}{c\lambda_n + d}. \tag{4.4}$$

Lemma

(b) If $\lambda_k \neq -\frac{d}{c} = \lambda_n$, then

$$(y_{k+2}, y_n) = -(ad - bc)\left(\frac{y_{k+2}(1)}{c\lambda_k + d} - \frac{c\, y_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right) \cdot \frac{y_n'(1)}{a\lambda_n + b}. \tag{4.5}$$

(c) If $\lambda_k = -\frac{d}{c} \neq \lambda_n$, then

$$(y_{k+2}, y_n) = -(ad - bc)\left(\frac{y_{k+2}'(1)}{a\lambda_k + b} - \frac{a\, y_{k+1}'(1)}{(a\lambda_k + b)^2} + \frac{a^2 y_k'(1)}{(a\lambda_k + b)^3}\right) \cdot \frac{y_n(1)}{c\lambda_n + d}. \tag{4.6}$$

Lemma

Lemma

Suppose that λ_k is an eigenvalue of multiplicity three ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$).

(a) If $\lambda_k \neq -\frac{d}{c}$, then

$$(y_{k+1}, y_k) = -(ad - bc)\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{c\, y_k(1)}{(c\lambda_k + d)^2}\right) \cdot \frac{y_k(1)}{c\lambda_k + d}. \tag{4.7}$$

(b) If $\lambda_k = -\frac{d}{c}$, then

$$(y_{k+1}, y_k) = -(ad - bc)\left(\frac{y'_{k+1}(1)}{a\lambda_k + b} - \frac{a\, y'_k(1)}{(a\lambda_k + b)^2}\right) \cdot \frac{y'_k(1)}{a\lambda_k + b}. \tag{4.8}$$

Lemma (part a)

Lemma

Suppose that λ_k is an eigenvalue of multiplicity three $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$.

(a) If $\lambda_k \neq -\frac{d}{c}$, then

$$(y_{k+2}, y_k) = -(ad - bc)\left(\frac{y_{k+2}(1)}{c\lambda_k + d} - \frac{c\, y_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right) \cdot \frac{y_k(1)}{c\lambda_k + d} + Q_k. \tag{4.9}$$

where

$$Q_k = \|y_{k+1}\|_2^2 + (ad - bc)\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{c\, y_k(1)}{(c\lambda_k + d)^2}\right)^2 \neq 0. \tag{4.10}$$

Lemma (part b)

Lemma (continued)

(b) If $\lambda_k = -\frac{d}{c}$, *then*

$$(y_{k+2}, y_k) = -(ad - bc)\left(\frac{y'_{k+2}(1)}{a\lambda_k + b} - \frac{a\,y'_{k+1}(1)}{(a\lambda_k + b)^2} + \frac{a^2 y'_k(1)}{(a\lambda_k + b)^3}\right) \cdot \frac{y'_k(1)}{a\lambda_k + b} + P_k. \qquad (4.11)$$

where

$$P_k = \|y_{k+1}\|_2^2 + (ad - bc)\left(\frac{y'_{k+1}(1)}{a\lambda_k + b} - \frac{a\,y'_k(1)}{(a\lambda_k + b)^2}\right)^2 \neq 0. \qquad (4.12)$$

Existence of special associated functions y^*_{k+1} and $y^{\#}_{k+1}$.

Lemma

Suppose that $\lambda_k \neq -\frac{d}{c}$ is an eigenvalue of multiplicity two. Then there exists an associated function

$$y^*_{k+1} = y_{k+1} + C_1 y_k,$$

with C_1 being a constant, such that

$$(y^*_{k+1}, y_{k+1}) = -(ad - bc)\left(\frac{y^*_{k+1}(1)}{c\lambda_k + d} - \frac{cy_k(1)}{(c\lambda_k + d)^2}\right)$$
$$\left(\frac{y_{k+1}(1)}{c\lambda_k + d} - \frac{cy_k(1)}{(c\lambda_k + d)^2}\right). \qquad (5.1)$$

Lemma

Suppose that $\lambda_k = -\frac{d}{c}$ *is an eigenvalue of multiplicity two. Then there exists an associated function*

$$y^*_{k+1} = y_{k+1} + C_1 y_k,$$

with C_1 *being a constant, such that*

$$(y^*_{k+1}, y_{k+1}) = -(ad - bc)\left(\frac{(y^*_{k+1})'(1)}{a\lambda_k + b} - \frac{a y'_k(1)}{(a\lambda_k + b)^2}\right)$$

$$\left(\frac{y'_{k+1}(1)}{a\lambda_k + b} - \frac{a y'_k(1)}{(a\lambda_k + b)^2}\right). \qquad (5.4)$$

Lemma

Suppose that $\lambda_k \neq -\frac{d}{c}$ is an eigenvalue of multiplicity three. Then there exists an associated function $y^{\#}_{k+1} = y_{k+1} + C_2 y_k$, with C_2 being a constant, such that

$$(y^{\#}_{k+1}, y_{k+2}) = -(ad - bc)\left(\frac{y^{\#}_{k+1}(1)}{c\lambda_k + d} - \frac{c y_k(1)}{(c\lambda_k + d)^2}\right) \times$$

$$\times \left(\frac{y_{k+2}(1)}{c\lambda_k + d} - \frac{c y_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right). \qquad (5.7)$$

Lemma

Suppose that $\lambda_k = -\frac{d}{c}$ *is an eigenvalue of multiplicity three. Then there exists an associated function* $y^{\#}_{k+1} = y_{k+1} + C_2 y_k$, *with* C_2 *being a constant, such that*

$$(y^{\#}_{k+1}, y_{k+2}) = -(ad - bc)\left(\frac{(y^{\#}_{k+1})'(1)}{a\lambda_k + b} - \frac{a y_k'(1)}{(a\lambda_k + b)^2}\right) \times$$

$$\times \left(\frac{y_{k+2}'(1)}{a\lambda_k + b} - \frac{a y_{k+1}'(1)}{(a\lambda_k + b)^2} + \frac{a^2 y_k'(1)}{(a\lambda_k + b)^3}\right). \qquad (5.9)$$

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Existence of special associated functions y^*_{k+2} and $y^{\#}_{k+2}$.

Observe that the function y^*_{k+2}, given by $y^*_{k+2} = y_{k+2} + C_2 y_{k+1}$, with the same constant C_2 satisfies all the lemmas for y_{k+2}

Lemma

Suppose that $\lambda_k \neq -\frac{d}{c}$ *is an eigenvalue of multiplicity three. Then there exists an associated function* $y^{\#}_{k+2} = y^{*}_{k+2} + D_1 y_k$*, with* D_1 *being a constant, such that*

$$(y^{\#}_{k+2}, y_{k+2}) = -(ad-bc)\left(\frac{y^{\#}_{k+2}(1)}{c\lambda_k + d} - \frac{c y^{\#}_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right)\times$$

$$\times\left(\frac{y_{k+2}(1)}{c\lambda_k + d} - \frac{c y_{k+1}(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}\right). \qquad (6.8)$$

Lemma

Lemma

Suppose that $\lambda_k = -\frac{d}{c}$ *is an eigenvalue of multiplicity three. Then there exists an associated function*

$$y_{k+2}^{\#} = y_{k+2}^{*} + D_1 y_k,$$

with D_1 *being a constant, such that*

$$\begin{aligned}(y_{k+2}^{\#}, y_{k+2}) = -(ad-bc)&\left(\frac{(y_{k+2}^{\#})'(1)}{a\lambda_k+b} - \frac{a\,(y_{k+1}^{\#})'(1)}{(a\lambda_k+b)^2} + \frac{a^2 y_k'(1)}{(a\lambda_k+b)^3}\right)\\ &\times\left(\frac{y_{k+2}'(1)}{a\lambda_k+b} - \frac{a\,y_{k+1}'(1)}{(a\lambda_k+b)^2} + \frac{a^2 y_k'(1)}{(a\lambda_k+b)^3}\right).\end{aligned} \tag{6.10}$$

Let us introduce the following notations:

$$\mathfrak{A}(y_n) = \begin{cases} \dfrac{y_n(1)}{c\lambda_n + d}, & \lambda_n \neq -\dfrac{d}{c}, \\ \dfrac{y_n'(1)}{a\lambda_n + b}, & \lambda_n = -\dfrac{d}{c}. \end{cases} \tag{7.1}$$

Similarly,

$$\mathfrak{A}(y_{k+1}) = \begin{cases} \dfrac{y_{k+1}(1)}{c\lambda_k + d} - \dfrac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{y_{k+1}'(1)}{a\lambda_k + b} - \dfrac{a y_k'(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\dfrac{d}{c}, \end{cases} \tag{7.2}$$

$$\mathfrak{A}(y_{k+2}) = \begin{cases} \dfrac{y_{k+2}(1)}{c\lambda_k + d} - \dfrac{c y_{k+1}(1)}{(c\lambda_k + d)^2} + \dfrac{c^2 y_k(1)}{(c\lambda_k + d)^3}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{y'_{k+2}(1)}{a\lambda_k + b} - \dfrac{a y'_{k+1}(1)}{(a\lambda_k + b)^2} + \dfrac{a^2 y'_k(1)}{(a\lambda_k + b)^3}, & \lambda_k = -\dfrac{d}{c}. \end{cases} \tag{7.3}$$

In particular,

$$\mathfrak{A}(y^*_{k+1}) = \begin{cases} \dfrac{y^*_{k+1}(1)}{c\lambda_k + d} - \dfrac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y^*_{k+1})'(1)}{a\lambda_k + b} - \dfrac{a y'_k(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\dfrac{d}{c}, \end{cases} \tag{7.4}$$

$$\mathfrak{A}(y_{k+1}^{\#}) = \begin{cases} \dfrac{y_{k+1}^{\#}(1)}{c\lambda_k + d} - \dfrac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y_{k+1}^{\#})'(1)}{a\lambda_k + b} - \dfrac{a y_k'(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\dfrac{d}{c}, \end{cases} \tag{7.5}$$

$$\mathfrak{A}(y_{k+2}^{*}) = \begin{cases} \dfrac{y_{k+2}^{*}(1)}{c\lambda_k + d} - \dfrac{c y_{k+1}^{\#}(1)}{(c\lambda_k + d)^2} + \dfrac{c^2 y_k(1)}{(c\lambda_k + d)^3}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y_{k+2}^{*})'(1)}{a\lambda_k + b} - \dfrac{a (y_{k+1}^{\#})'(1)}{(a\lambda_k + b)^2} + \dfrac{a^2 y_k'(1)}{(a\lambda_k + b)^3}, & \lambda_k = -\dfrac{d}{c}, \end{cases} \tag{7.6}$$

$$\mathfrak{A}(y^{\#}_{k+2}) = \begin{cases} \dfrac{y^{\#}_{k+2}(1)}{c\lambda_k + d} - \dfrac{c y^{\#}_{k+1}(1)}{(c\lambda_k + d)^2} + \dfrac{c^2 y_k(1)}{(c\lambda_k + d)^3}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y^{\#}_{k+2})'(1)}{a\lambda_k + b} - \dfrac{a (y^{\#}_{k+1})'(1)}{(a\lambda_k + b)^2} + \dfrac{a^2 y'_k(1)}{(a\lambda_k + b)^3}, & \lambda_k = -\dfrac{d}{c}. \end{cases} \tag{7.7}$$

Case (i).

Theorem

Assume that all eigenvalues of problem (1.1)–(1.3) are real and simple. Then the system

$$\{y_n\} \quad (n = 0, 1, \ldots;\ n \neq l), \tag{7.8}$$

where l is a nonnegative integer, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

The elements of the system *(7.9)* by

$$u_n(x) = \frac{1}{B_n \mathfrak{A}(y_l)} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}. \tag{7.10}$$

Case (ii).

Theorem

If λ_k is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \quad (n = 0, 1, \dots;\ n \neq k+1), \tag{7.12}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

The biorthogonal system in this case is defined by

$$u_n(x) = \frac{1}{B_n \mathfrak{A}(y_k)} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y_k(x) & \mathfrak{A}(y_k) \end{vmatrix} \ (n \neq k,\ k+1), \tag{7.13}$$

$$u_k(x) = \frac{1}{\beta_k \mathfrak{A}(y_k)} \begin{vmatrix} y_{k+1}(x) & \mathfrak{A}(y_{k+1}) \\ y_k(x) & \mathfrak{A}(y_k) \end{vmatrix}, \tag{7.14}$$

where $\beta_k = T_k$ if $\lambda_k \neq -\frac{d}{c}$, and $\beta_k = S_k$ if $\lambda_k = -\frac{d}{c}$.

Theorem

If λ_k is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \ (n = 0, \ 1, \dots ; n \neq k), \tag{7.15}$$

is a basis of the space $L_p(0, 1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^*_{k+1}) \neq 0$.

The biorthogonal system is defined as

$$u_n(x) = \frac{1}{B_n \mathfrak{A}(y^*_{k+1})} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y^*_{k+1}(x) & \mathfrak{A}(y^*_{k+1}) \end{vmatrix} \ (n \neq k, \ k+1), \tag{7.16}$$

$$u_{k+1}(x) = \frac{1}{\beta_k \mathfrak{A}(y^*_{k+1})} \begin{vmatrix} y_k(x) & \mathfrak{A}(y_k) \\ y^*_{k+1}(x) & \mathfrak{A}(y^*_{k+1}) \end{vmatrix}. \tag{7.17}$$

Theorem

If λ_k is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \ (n = 0,\ 1, \ldots ; n \neq l), \tag{7.18}$$

where $l \neq k,\ k+1$ is a non-negative integer, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Proof.

The biorthogonal system coincides with the expression given in (7.10) for $n \neq k, k+1$, and

$$u_k(x) = \frac{1}{\beta_k \mathfrak{A}(y_l)} \begin{vmatrix} y^*_{k+1}(x) & \mathfrak{A}(y^*_{k+1}) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}, \tag{7.19}$$

$$u_{k+1}(x) = \frac{1}{\beta_k \mathfrak{A}(y_l)} \begin{vmatrix} y_k(x) & \mathfrak{A}(y_k) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}. \tag{7.20}$$

$\square$

Case (iii).

Theorem

If λ_k is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \ \ (n = 0,\ 1, \dots ; n \neq k+2), \tag{8.1}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$

Proof.

The biorthogonal system is given by the formula (7.13) for $n \neq k,\ \ k+1,\ \ k+2$, and

$$u_k(x) = \frac{1}{\gamma_k \mathfrak{A}(y_k)} \begin{vmatrix} y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \\ y_k(x) & \mathfrak{A}(y_k) \end{vmatrix}, \tag{8.2}$$

$$u_{k+1}(x) = \frac{1}{\gamma_k \mathfrak{A}(y_k)} \begin{vmatrix} y_{k+1}(x) & \mathfrak{A}(y_{k+1}) \\ y_k(x) & \mathfrak{A}(y_k) \end{vmatrix}. \tag{8.3}$$

where $\gamma_k = Q_k$ if $\lambda_k \neq -\frac{d}{c}$, and $\gamma_k = P_k$ if $\lambda_k = -\frac{d}{c}$. ☐

Theorem

If λ_k is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \ \ (n = 0,\ 1, \ldots ; n \neq k+1), \tag{8.4}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^{\#}_{k+1}) \neq 0$.

Proof.

The biorthogonal system is defined by

$$u_n(x) = \frac{1}{B_n \mathfrak{A}(y^{\#}_{k+1})} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y^{\#}_{k+1}(x) & \mathfrak{A}(y^{\#}_{k+1}) \end{vmatrix} \ \ (n \neq k,\ k+1,\ k+2), \tag{8.5}$$

$$u_k(x) = \frac{1}{\gamma_k \mathfrak{A}(y^{\#}_{k+1})} \begin{vmatrix} y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \\ y^{\#}_{k+1}(x) & \mathfrak{A}(y^{\#}_{k+1}) \end{vmatrix}, \tag{8.6}$$

$$u_{k+2}(x) = \frac{1}{\gamma_k \mathfrak{A}(y^{\#}_{k+1})} \begin{vmatrix} y_k(x) & \mathfrak{A}(y_k) \\ y^{\#}_{k+1}(x) & \mathfrak{A}(y^{\#}_{k+1}) \end{vmatrix}. \tag{8.7}$$

Theorem

If λ_k is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \ (n = 0, \ 1, \dots ; n \neq k), \tag{8.8}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^{\#}_{k+2}) \neq 0$.

Proof.

We define the elements of the biorthogonal system by

$$u_n(x) = \frac{1}{B_n \mathfrak{A}(y^{\#}_{k+2})} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \end{vmatrix} \ (n \neq k, \ k+1, \ k+2), \tag{8.9}$$

$$u_{k+1}(x) = \frac{1}{\gamma_k \mathfrak{A}(y^{\#}_{k+2})} \begin{vmatrix} y_{k+1}(x) & \mathfrak{A}(y_{k+1}) \\ y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \end{vmatrix}, \tag{8.10}$$

$$u_{k+2}(x) = \frac{1}{\gamma_k \mathfrak{A}(y^{\#}_{k+2})} \begin{vmatrix} y_k(x) & \mathfrak{A}(y_k) \\ y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \end{vmatrix}. \tag{8.11}$$

Theorem

If λ_k is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \;\; (n = 0, \; 1, \ldots; n \neq l), \tag{8.12}$$

where $l \neq k, \;\; k+1, \;\; k+2$ is a non-negative integer, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Proof.

The biorthogonal system is defined by the formula (7.10) for $n \neq k, \;\; k+1, \;\; k+2, \;\; l$, and

$$u_k(x) = \frac{1}{\gamma_k \mathfrak{A} y_l} \begin{vmatrix} y^{\#}_{k+2}(x) & \mathfrak{A}(y^{\#}_{k+2}) \\ y_l(x) & \mathfrak{A} y_l \end{vmatrix}. \tag{8.13}$$

$$u_{k+1}(x) = \frac{1}{\gamma_k \mathfrak{A}(y_l)} \begin{vmatrix} y^{\#}_{k+1}(x) & \mathfrak{A}(y^{\#}_{k+1}) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}, \tag{8.14}$$

$$u_{k+2}(x) = \frac{1}{\gamma_k \mathfrak{A}(y_l)} \begin{vmatrix} y_k(x) & \mathfrak{A}(y_k) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}, \tag{8.15}$$

Case (iv).

Theorem

Let λ_r and $\lambda_s = \overline{\lambda_r}$ be a pair of complex conjugate non-real eigenvalues. Then each of the systems

$$\{y_n\} \quad (n = 0, 1, \dots; \ n \neq r), \tag{8.16}$$

and

$$\{y_n\} \quad (n = 0, 1, \dots; \ n \neq l), \tag{8.17}$$

where l is a non-negative integer with $l \neq r, s$, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Proof.

For (8.16) the biorthogonal system is

$$u_n(x) = \frac{1}{B_n\mathfrak{A}(y_s)} \begin{vmatrix} y_n(x) & \mathfrak{A}(y_n) \\ y_s(x) & \mathfrak{A}(y_s) \end{vmatrix} \ (n \neq r,\ s), \tag{8.18}$$

$$u_s(x) = \frac{1}{T_r\mathfrak{A}(y_s)} \begin{vmatrix} y_r(x) & \mathfrak{A}(y_r) \\ y_s(x) & \mathfrak{A}(y_s) \end{vmatrix}. \tag{8.19}$$

The biorthogonal system of (8.17) is defined by (7.10) for $n \neq r, s, l$ and

$$u_r(x) = \frac{1}{T_s\mathfrak{A}(y_l)} \begin{vmatrix} y_s(x) & \mathfrak{A}(y_s) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}. \tag{8.20}$$

$$u_s(x) = \frac{1}{T_r\mathfrak{A}(y_l)} \begin{vmatrix} y_r(x) & \mathfrak{A}(y_r) \\ y_l(x) & \mathfrak{A}(y_l) \end{vmatrix}. \tag{8.21}$$

□

Example 1.

$$-y'' = \lambda y,\ 0 < x < 1,$$

$$y'(0) = 0,\ 3\lambda y(1) = (\lambda - 3)\, y'(1).$$

Note that $a = 3,\ b = 0,\ c = 1,\ d = -3,\ ad - bc = -9 < 0$, $q(x) \equiv 0,\ \lambda_0 = \lambda_1 = \lambda_2 = 0$ is the eigenvalue of multiplicity three, and $\lambda_0 \neq -\frac{d}{c}$. The remaining eigenvalues $\lambda_3 < \lambda_4 < \ldots$ are the positive zeros of the transcendental function

$$\omega(\lambda) = 3\lambda \cos\sqrt{\lambda} + (\lambda - 3)\sqrt{\lambda}\sin\sqrt{\lambda}.$$

The eigenfunctions are $y_0 = 1,\ y_n = \cos\sqrt{\lambda_n}x\ (n \geq 3)$ and the first associated function of y_0 is $y_1 = -\frac{1}{2}x^2 + C$, where C is a constant. Furthermore, the second associated function of y_0 is $y_2 = \frac{1}{24}x^4 - \frac{C}{2}x^2 + D$.

Example 2.

$$-y'' = \lambda y,\ 0 < x < 1,$$
$$y(0) = -y'(0),\ (9\lambda + 15)y(1) = 5\lambda y'(1).$$

Note that now $a = 9,\ b = 15,\ c = 5,\ d = 0,\ ad - bc = -75 < 0,$ again $q(x) \equiv 0,\ \lambda_0 = \lambda_1 = \lambda_2 = 0$ is the eigenvalue of multiplicity three, and $\lambda_0 = -\frac{d}{c}$. The other eigenvalues $\lambda_3 < \lambda_4 < \dots$ are the positive zeros of

$$\omega(\lambda) = (9\lambda + 15)\left(\cos\sqrt{\lambda} - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right) + 5\lambda\left(\sqrt{\lambda}\sin\sqrt{\lambda} + \cos\sqrt{\lambda}\right).$$

The eigenfunctions are $y_0 = 1 - x,\ y_n = \cos\sqrt{\lambda_n}x - \frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}$ $(n \geq 3)$, and the associated functions are $y_1 = \frac{x^3}{6} - \frac{x^2}{2} + C\cdot(x-1)$ and $y_2 = -\frac{x^5}{120} + \frac{x^4}{24} + C\cdot\left(-\frac{x^3}{6} + \frac{x^2}{2}\right) + D\cdot(x-1)$.