# A Class of Functionals on the Sequence Space $s$ Satisfying the Palais-Smale Condition

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# Motivation: Nonlinear Operator Problems

A classical approach to solving nonlinear operator problems in function spaces (PDEs, spectral problems, and etc) is the variational formulation:

$$\varphi'(x) = 0,$$

where $\varphi$ is a functional with some regularity assumptions over Hilbert, Sobolov, Banach, or Fréchet spaces.

This framework relies on the compactness of the functional. Specifically, the existence of a convergent subsequence for any Palais–Smale sequence $(x_n)$ satisfying:

$$\varphi(x_n) \text{ is bounded }, \quad \varphi'(x_n) \to 0.$$

# Beyond Banach Spaces: Main Objectives

Beyond Banach spaces, the Palais–Smale condition is difficult to obtain and verify in practice.

**Our Goal & Framework:**

- Introduce a class of functionals on the Fréchet space $s$ of rapidly decreasing sequences that satisfy this condition.
- Formulate specific nonlinear operator problems as minimization problems of these functionals. Establish the existence, uniqueness, and regularity of their solutions.

# Reasons behind the Choice of $s$

1. **Topological Advantage: The Montel Property**

   - The space $s$ is a Montel space; it satisfies the Heine–Borel property: a subset is compact if and only if it is closed and bounded. This characterization significantly facilitates verifying the Palais–Smale condition, as compactness criteria reduce to establishing boundedness.

2. **Structural Universality: Kōmura–Kōmura Embedding**

   - The Kōmura–Kōmura embedding theorem: every nuclear locally convex space is isomorphic to a linear subspace of $s^{\mathbb{N}}$; including
     - The Schwartz space $\mathcal{S}(\mathbb{R})$
     - Compactly supported smooth functions $\mathscr{D}[a, b]$
     - Periodic smooth functions $C^{\infty}_{2\pi}(\mathbb{R})$ and $C^{\infty}[a, b]$

# The Sequence Space $s$ and its Dual $\mathrm{t}$

## The Space $s$ of Rapidly Decreasing Sequences

$$s := \left\{ x = (x_n) \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \forall k \in \mathbb{N}_0, \quad \|x\|_{s,k} := \sup_n |x_n| n^k < \infty \right\}$$

The space $s$ is Fréchet–Montel.

## The Strong Dual $\mathrm{t} := s'$ of Tempered Sequences

$$\mathrm{t} := \bigcup_{k \in \mathbb{N}_0} \left\{ y = (y_n) \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \|y\|_{\mathrm{t},k} := \sup_n |y_n| n^{-k} < \infty \right\}$$

- The strong topology is the inductive limit topology, the space $\mathrm{t}$ is a DF-space and a Montel space.
- The strong topology coincides with the topology of compact convergence generated by the seminorms:

$$\|\ell\|_{\mathrm{t},K} := \sup_{h \in K} |\ell(h)|, \quad \text{where } K \subset s \text{ is compact.}$$

# Keller's $C_c^k$-Differentiability

Let E and F be Fréchet spaces, and let $CL(\mathrm{E}, \mathrm{F})$ be the space of all continuous linear mappings endowed with the compact convergence topology.

**Definition (Keller, 1974)**

Let $\varphi : U \overset{open}{\subseteq} \mathrm{E} \to \mathrm{F}$ be a map. We say that $\varphi$ is a Keller-differentiable map of class $C_c^1$ if:

1. The directional derivatives

$$\mathrm{D}\varphi(x)h = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t}$$

   exist for all $x \in U$ and all $h \in \mathrm{E}$.

2. The induced map $\mathrm{D}\varphi : U \to CL(\mathrm{E}, \mathrm{F})$ is continuous.

Higher directional derivatives and $C_c^k$-mappings $(k \geqslant 2)$ are defined inductively in the usual manner.

# The Palais–Smale Condition in Fréchet Spaces

**Definition**
Let $\varphi\colon \mathsf{F} \to \mathbb{R}$ be a Keller-differentiable functional of class $C_c^1$.

(i) We say that $\varphi$ satisfies the Palais–Smale condition (PS condition) if every sequence $(x_i) \subset \mathsf{F}$ for which $(\varphi(x_i))$ is bounded and

$$\mathrm{D}\varphi(x_i) \xrightarrow{\mathsf{F}'_c} 0$$

has a convergent subsequence.

(ii) We say that $\varphi$ satisfies the Palais–Smale condition at level $m$ ($(\mathrm{PS})_m$ condition) if every sequence $(x_i) \subset \mathsf{F}$ for which

$$\varphi(x_i) \to m \quad \text{and} \quad \mathrm{D}\varphi(x_i) \xrightarrow{\mathsf{F}'_c} 0$$

has a convergent subsequence.

**Proposition**
The Palais–Smale condition is preserved under linear homeomorphisms.

# The Class $\mathcal{F}_s$ and Associated Functionals

**Definition (Class $\mathcal{F}_s$)**
Let $\mathcal{F}_s$ denote the class of sequence pairs $(a_n, f_n)_{n\in\mathbb{N}}$ satisfying:

**A.1 Condition on $a_n$:** There exist constants $\alpha, M > 0$ such that

$$0 < \alpha \leqslant a_n \leqslant M \quad \forall n \in \mathbb{N}.$$

**A.1 Conditions on $f_n$:** Each $f_n \in C^1(\mathbb{R})$ is convex and satisfies:

- Quadratic growth: $|f_n(t)| \leqslant \beta_n(1 + t^2)$ for all $t \in \mathbb{R}$, where $(\beta_n) \in s$.
- Lower bound: $f_n(t) \geqslant -\gamma_n$ for all $t \in \mathbb{R}$, where $\sum_{n=1}^{\infty} \gamma_n < \infty$.

**Definition ($\mathcal{F}_s$-Functional)**
Given a pair $(a_n, f_n) \in \mathcal{F}_s$, the associated functional $\varphi \colon s \to \mathbb{R}$ evaluated at a sequence $x = (x_n)$ is defined by:

$$\varphi(x) := \frac{1}{2} \sum_{n=1}^{\infty} a_n {x_n}^2 + \sum_{n=1}^{\infty} f_n(x_n)$$

# Main Results: Differentiability and Solvability

**Lemma**
*Every $\mathcal{F}_s$-functional $\varphi\colon s \to \mathbb{R}$ is a Keller-differentiable mapping of class $C_c^1$.*

**Theorem**
*Every $\mathcal{F}_s$-functional $\varphi\colon s \to \mathbb{R}$ satisfies the Palais–Smale condition.*

**Corollary (Existence & Uniqueness)**
Every $\mathcal{F}_s$-functional $\varphi\colon s \to \mathbb{R}$ admits a unique global minimum on $s$.

## Sketch of the Proof

Consider any PS-sequence $(x^{(j)})$, we prove $\sup_n |x_n^{(j)}| n^k < \infty$.

1. **$\ell^2$-norm Bound:**
   Applying the uniform lower bounds $a_n \geqslant \alpha > 0$ and $f_n(t) \geqslant -\gamma_n$ with $\sum \gamma_n < \infty$, the bounded condition $\varphi(x^{(j)}) \leqslant N$ establishes a uniform bound in the $\ell^2$-norm:
   $$\|x^{(j)}\|_{\ell^2}^2 \leqslant \frac{2}{\alpha}(N + C_\gamma), \quad \text{where } C_\gamma := \sum_{n=1}^{\infty} \gamma_n.$$

2. **Boundedness in $s$**
   The second condition $g^{(j)} := \mathrm{D}\varphi(x^{(j)}) \xrightarrow{\mathrm{t}} 0$ is evaluated componentwise for an arbitrary index $k \in \mathbb{N}$:
   - **Algebraic Identity:** Rearrange the relation to $a_n x_n^{(j)} = g_n^{(j)} - f_n'(x_n^{(j)})$ to construct the series $\sum_{n=1}^{\infty} a_n (x_n^{(j)})^2 n^{2k}$. This leads to the central inequality
   $$\alpha \sum_{n=1}^{\infty} (x_n^{(j)})^2 n^{2k} \leqslant \left| \sum_{n=1}^{\infty} g_n^{(j)} x_n^{(j)} n^{2k} \right| + \left| \sum_{n=1}^{\infty} f_n'(x_n^{(j)}) x_n^{(j)} n^{2k} \right|.$$

# Sketch of the Proof

1. **Boundedness in $s$**
   - **Perturbation Estimate:** Apply the linear growth bound $|f_n'(t)| \leqslant \widehat{\beta}_n(1+|t|)$, where $(\widehat{\beta}_n) \in s$. Majorize this summation via the Cauchy–Schwarz inequality and the prior $\ell^2$-norm bound to establish a uniform bound $C_k'$ independent of $j$.
   - **Absorption via Young's Inequality:** Parametrize Young's inequality with $\varepsilon = \alpha$ on the dual term to isolate the higher-order coordinate components on the left-hand side of the estimate:

$$\left|\sum_{n=1}^{\infty} g_n^{(j)} x_n^{(j)} n^{2k}\right| \leqslant \sum_{n=1}^{\infty} |g_n^{(j)} n^k||x_n^{(j)} n^k| \leqslant \frac{1}{2\alpha}\sum_{n=1}^{\infty}(g_n^{(j)})^2 n^{2k} + \frac{\alpha}{2}\sum_{n=1}^{\infty}(x_n^{(j)})^2 n^{2k}.$$

$$\leqslant C_k'' + C_k'$$

   for some constant $C_k''$ for sufficiently large $j$, yielding:

$$\sup_n |x_n^{(j)}| n^k < \infty.$$

# Roles of the Base Weights $a_n$

| Assumption | Located In | Roles |
| --- | --- | --- |
| $a_n \leqslant M$ | Well-definedness & Differentiability | Enables majorization of the series $\sum a_n x_n^2$, $\sum a_n x_n h_n$, and $\sum a_n h_n^2$ by convergent $p$-series, and bounds the linear component of the derivative map $\mathrm{D}\varphi$. |
| $a_n \geqslant \alpha > 0$ | PS Condition | Establishes a uniform bound in the $\ell^2$-norm via the estimate $\|x^{(j)}\|_{\ell^2}^2 \leqslant \frac{2}{\alpha}(N + C_\gamma)$, and parametrizes Young's inequality to isolate higher-order seminorms. |

# Roles of the Functions $f_n$

| Assumption | Located In | Roles |
|---|---|---|
| $f_n \geqslant -\gamma_n$ | PS & Global Min | Guarantees that the functional is bounded from below ($\varphi \geqslant -C_\gamma$) via the convergence of $\sum \gamma_n$. |
| $\lvert f_n(t)\rvert \leqslant \beta_n(1+t^2)$ , $(\beta_n) \in s$ | Differentiability & PS Condition | Yields the derivative growth coefficients ($\widehat{\beta}_n = 7\beta_n$) inherit the rapid decay of $s$, justifying the difference quotient limit and the sequential continuity of $\mathrm{D}\varphi\colon s \to$ t via uniform convergence. |
| **Convexity** | Differentiability & Global Min | Imposes monotonicity on $f_n'$, enabling local Mean Value Theorem estimates to extend to the global linear growth bounds required for the existence and continuity of $\mathrm{D}\varphi$. Ensures strict convexity of the total functional. |

# The Space $C^\infty_{2\pi}(\mathbb{R})$

Let $C^\infty_{2\pi}(\mathbb{R})$ be the Fréchet space of all smooth, $2\pi$-periodic, real-valued functions. This space is isomorphic to the sequence space $s$.

- Every function $f \in C^\infty_{2\pi}(\mathbb{R})$ corresponds uniquely to its Fourier series expansion:

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(nx) + \beta_n \sin(nx))$$

where the Fourier coefficients are:

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad \beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

**The Isometric Isomorphism**

The mapping $L\colon C^\infty_{2\pi}(\mathbb{R}) \to s$ constructs a linear homeomorphism by ordering the coefficients into a single real sequence $x = (x_k)_{k\in\mathbb{N}} \in s$:

$$x_1 = \alpha_0, \quad x_{2n} = \alpha_n, \quad x_{2n+1} = \beta_n \quad (\text{for } n \geqslant 1)$$

# Induced Functional on $C^\infty_{2\pi}(\mathbb{R})$

Le $(a_n, f_n) \in \mathcal{F}_s$ and $\varphi\colon s \to \mathbb{R}$ be its associated functional. We define the pull-back functional $G\colon C^\infty_{2\pi}(\mathbb{R}) \to \mathbb{R}$ via composition with the linear homeomorphism $L$:

$$G(f) := \varphi(L(f))$$

Explicitly substituting the Fourier coefficients into the definition of $\varphi$ yields:

$$G(f) = \frac{1}{2}\left(a_1\alpha_0^2 + \sum_{n=1}^{\infty}(a_{2n}\alpha_n^2 + a_{2n+1}\beta_n^2)\right) + \left(f_1(\alpha_0) + \sum_{n=1}^{\infty}(f_{2n}(\alpha_n) + f_{2n+1}(\beta_n))\right)$$

The functional $G$ satisfies the Palais–Smale condition.

# The Schwartz Space $\mathcal{S}(\mathbb{R})$

Let $\mathcal{S}(\mathbb{R})$ denote the Fréchet space of rapidly decreasing smooth functions, topologized by the family of seminorms:

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \|f\|_k^2 := \sum_{\alpha+\beta \leqslant k} \int_{\mathbb{R}} |x|^{2\alpha} |f^{(\beta)}(x)|^2 \, dx < \infty \ \forall k \in \mathbb{N} \right\}$$

Let $(H_n)_{n\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ denote the sequence of Hermite functions, which constitutes an orthonormal basis for $L^2(\mathbb{R})$.

- The Hermite transform $H \colon \mathcal{S}(\mathbb{R}) \to s$ is defined by mapping each function $f$ to its sequence of expansion coefficients:

$$H(f) := \big(\langle f, H_{k-1} \rangle\big)_{k\in\mathbb{N}}$$

- The operator $H$ is a linear homeomorphism between $\mathcal{S}(\mathbb{R})$ and $s$.

# Induced Functional

**The Pullback Functional $G$**

- Let $\varphi\colon s \to \mathbb{R}$ be an $\mathcal{F}_s$-functional. We define the pullback functional $G\colon \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ via composition with the linear homeomorphism $H$:

$$G(f) := \varphi(H(f))$$

- Explicit evaluation yields the coordinate representation:

$$G(f) = \frac{1}{2}\sum_{k=1}^{\infty} a_k \big\langle f, H_{k-1}\big\rangle^2 + \sum_{k=1}^{\infty} f_k\big(\langle f, H_{k-1}\rangle\big)$$

The functional $G$ satisfies the Palais–Smale condition.

# The Space of Compactly Supported Functions $\mathscr{D}[a,b]$

Let $\mathscr{D}[a,b]$ denote the Fréchet space of smooth functions on $\mathbb{R}$ whose support is compactly contained within the bounded interval $[a,b]$:

$$\mathscr{D}[a,b] := \{\phi \in C^\infty(\mathbb{R}) : \operatorname{supp}(\phi) \subset [a,b]\}$$

The space $\mathscr{D}[a,b]$ is topologically isomorphic to the space $s$. The linear homeomorphism $L^{-1}\colon \mathscr{D}[a,b] \to s$ maps each function $\phi$ to a sequence $x = (x_k)_{k\in\mathbb{N}} \in s$, defined via the integral transformation:

$$x_k(\phi) = \int_{\mathbb{R}} \phi\left(a + \frac{b-a}{\pi}\left(\arctan(\eta) + \frac{\pi}{2}\right)\right) H_{k-1}(\eta)\, d\eta$$

where $(H_n)_{n\in\mathbb{N}_0}$ represents the orthonormal sequence of Hermite functions in $L^2(\mathbb{R})$. The integration variable $\eta \in \mathbb{R}$ is the coordinate of the intermediate Schwartz space $\mathcal{S}(\mathbb{R})$ induced by the diffeomorphism pullback operator $\Phi^{-1}\colon \mathscr{D}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathcal{S}(\mathbb{R})$, defined by:

$$(\Phi^{-1}(g))(\eta) = g(\arctan(\eta)) \quad \text{for } g \in \mathscr{D}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

# Induced Functional

Let $F\colon s \to \mathbb{R}$ be an $\mathcal{F}_s$-functional. We define the induced pullback functional $G\colon \mathscr{D}[a,b] \to \mathbb{R}$ via composition with the coordinate mapping $L^{-1}$:

$$G(\phi) := F(L^{-1}(\phi))$$

Substitution of the coordinate sequences yields the explicit expansion:

$$G(\phi) = \frac{1}{2}\sum_{k=1}^{\infty} a_k \left(x_k(\phi)\right)^2 + \sum_{k=1}^{\infty} f_k \left(x_k(\phi)\right)$$

The functional $G$ satisfies the Palais–Smale condition.

# The Space of Smooth Functions on $C^\infty[a,b]$

Let $C^\infty[a,b]$ denote the space of all smooth, real-valued functions defined on the closed bounded interval $[a,b]$. This space is isomorphic space $s$.

**Chebyshev Coordinate Representation**

- For any function $\phi \in C^\infty[a,b]$, the corresponding coordinate sequence $x = (x_k(\phi))_{k\in\mathbb{N}} \in s$ is determined via weighted integration using the Chebyshev polynomials of the first kind, $(T_n(y))_{n\in\mathbb{N}_0}$:

$$x_1(\phi) = \frac{1}{\pi\sqrt{2\pi}} \int_{-1}^{1} \phi\left(a + \frac{b-a}{2}(y+1)\right) \frac{dy}{\sqrt{1-y^2}}$$

- and for higher-order modes ($k \geqslant 2$):

$$x_k(\phi) = \frac{2}{\pi\sqrt{2\pi}} \int_{-1}^{1} \phi\left(a + \frac{b-a}{2}(y+1)\right) T_{k-1}(y) \frac{dy}{\sqrt{1-y^2}}$$

# Induced Functional

**The Pullback Functional $G$**

- Let $F\colon s \to \mathbb{R}$ be an $\mathcal{F}_s$-functional. The induced functional $G\colon C^\infty[a,b] \to \mathbb{R}$ is defined via the coordinate assignment mapping $G(\phi) := F(x(\phi))$.
- Substituting the explicit coordinate integrals into the functional structure yields:

$$G(\phi) = \frac{1}{2}\sum_{k=1}^{\infty} a_k \left(x_k(\phi)\right)^2 + \sum_{k=1}^{\infty} f_k \left(x_k(\phi)\right)$$

The functional $G$ satisfies the Palais–Smale condition.

# Nonlinear Operator Equations: Problem Formulation

Consider the following nonlinear operator equation:

$$u(x) + \mathcal{K}(u(x)) + \mathcal{N}(u(x)) = f(x) \quad \text{in } L^2(0,\pi)$$

$L^2(0,\pi)$ is the Hilbert space with the standard orthonormal basis $\left\{\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$. The unknown function $u(x) = \sum_{n=1}^{\infty} x_n\phi_n(x)$.

- **Linear Operator:** $\mathcal{K}$ is a self-adjoint, compact operator acting diagonally on the basis elements with eigenvalues $\lambda_n = \frac{1}{n^2+1} \in s$.
- **Nonlinear Perturbation:** $\mathcal{N}$ is a Diagonal operator in basis defined by

$$\mathcal{N}(u(x)) = \sum_{n=1}^{\infty} \mu_n g(x_n)\phi_n(x), \quad \text{where } \mu_n \in s$$

  $\mu_n = \frac{1}{n!}$ and $g = \tanh$:

- **Source Data:** $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$ , $(c_n) \in s$.

# Nonlinear Operator Equations: Variational Formulation

- Projecting the equation onto the orthonormal basis via inner products transforms the operator problem into an infinite system of nonlinear equations for the coordinate sequence $x = (x_n)$:

$$\left(1 + \overset{\frac{1}{n^2+1}}{\lambda_n}\right) x_n + \overset{\frac{1}{n!}}{\mu_n}\, \overset{\tanh(x_n)}{g(x_n)} = c_n, \quad n = 1, 2, \dots.$$

- This system has no closed-form solutions.

- We define the functional $F\colon s \to \mathbb{R}$ corresponding to this coordinate system by:

$$F(x) = \sum_{n=1}^{\infty} \left( \frac{1}{2} \left(1 + \frac{1}{n^2+1}\right) x_n^2 + \frac{1}{n!} \log(\cosh(x_n)) - c_n x_n \right)$$

- The critical points of $F$ correspond precisely to the sequences solving the original nonlinear system.
- $F$ satisfies the structural conditions of an $\mathcal{F}_s$-functional, $(x_n^*) \in s$ is a unique global minimum, and $u^*(x) = \sum_{n=1}^{\infty} x_n^* \phi_n(x) \in C^{\infty}(0, \pi)$.

# Semilinear Elliptic PDEs: Problem Formulation

Consider the following boundary value problem for a semilinear elliptic PDE:

$$-\Delta u + g(u) = f(x) \quad \text{in } L^2(0,\pi) \tag{1}$$

subject to the Dirichlet boundary conditions $u(0) = u(\pi) = 0$.

- $u(x) = \sum_{n=1}^{\infty} x_n \phi_n(x)$.
- $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ with coefficients satisfying $(c_n) \in s$.
- $g(u)$ is assumed to act diagonally on the spectral coefficients (i.e., the $n$-th coefficient of $g(u)$ is given by $g(x_n)$), $g(t) = \tanh(t)$.
- The action of the negative Laplacian $-\Delta$ on the basis functions $\phi_n(x)$ has the eigenvalues $\lambda_n = n^2$, i.e., $-\Delta\phi_n = n^2\phi_n$.

# Semilinear Elliptic PDEs: Variational Formulation

Projecting Equation (1) onto the spectral eigenbasis transforms the differential equation into an infinite system of algebraic equations for the sequence $x = (x_n)$:

$$\overset{n^2}{\lambda_n} x_n + \overset{\tanh(x_n)}{g(x_n)} = c_n \quad \forall n \in \mathbb{N}$$

The system has no closed-form solutions.

We define the functional $F\colon s \to \mathbb{R}$ corresponding to this coordinate system by:

$$F(x) = \sum_{n=1}^{\infty} \left( \frac{1}{2} x_n^2 + \frac{1}{n^2} \ln(\cosh(x_n)) - \frac{1}{n^2} c_n x_n \right)$$

- The critical points of $F$ correspond exactly to the coordinate sequences solving the transformed semilinear system.

# Semilinear Elliptic PDEs: Variational Formulation

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- The critical points of $F$ correspond exactly to the coordinate sequences solving the transformed semilinear system.
- $F$ satisfies the structural conditions of an $\mathcal{F}_s$-functional, $(x_n^*) \in s$ is a unique global minimum, and $u^*(x) = \sum_{n=1}^{\infty} x_n^* \phi_n(x) \in C^\infty(0, \pi)$.

# Nonlinear Spectral Problems: Problem Formulation

We consider the problem of identifying an unknown function $f(t)$ within the Schwartz space $\mathcal{S}(\mathbb{R})$ that satisfies the following nonlinear spectral problem:

$$\sum_{n=1}^{\infty} \left(a_n \langle f, H_{n-1} \rangle + \nu_n h(\langle f, H_{n-1} \rangle) - c_n\right) H_{n-1}(t) = 0$$

- The Hermite functions: $(H_n(t))_{n \in \mathbb{N}_0}$
- $0 < \alpha \leqslant a_n \leqslant M$.
- **Nonlinear Terms:** $(\nu_n) \in s, \nu_n > 0$, $h = \arctan$.
- **Source Bounds:** $(c_n) \in s$. We impose a uniform ratio condition via a constant $M_{\max} < \pi/2$ s.t. $|c_n/\nu_n| \leqslant M_{\max}$ holds $\forall n \in \mathbb{N}$.

# Nonlinear Spectral Problems: Variational Formulation

By invoking the orthonormality of the Hermite basis, this spectral problem transforms into an infinite algebraic system for the sequence $x = (x_n) \in s$:

$$a_n x_n + \nu_n \overset{\arctan(x_n)}{h(x_n)} - c_n = 0 \quad \forall n \in \mathbb{N}$$

This system lacks a closed-form Solutions.

We define the functional $F : s \to \mathbb{R}$ corresponding to this coordinate system by:

$$F(x) = \frac{1}{2}\sum_{n=1}^{\infty} a_n x_n^2 + \sum_{n=1}^{\infty}\left(\nu_n\left(x_n \arctan(x_n) - \frac{1}{2}\log(1 + x_n^2)\right) - c_n x_n\right)$$

- The critical points of $F(x)$ match precisely with the sequence solutions of the underlying algebraic system.
- $F$ is an $\mathcal{F}_s$-functional, $x^* = (x_n^*) \in s$ is a unique minimum, $f^*(t) = \sum_{n=1}^{\infty} x_n^* H_{n-1}(t) \in \mathcal{S}(\mathbb{R})$,