Exponential Tail Estimates for Lacunary Trigonometric Series

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Outline

Lacunary trigonometric sums

$$f_N(x) = \sum_{k=1}^{N} c_k \sqrt{2} \cos(2\pi n_k x), \qquad x \in [0,1].$$

where n_k is a strictly increasing sequence of integers satisfying:

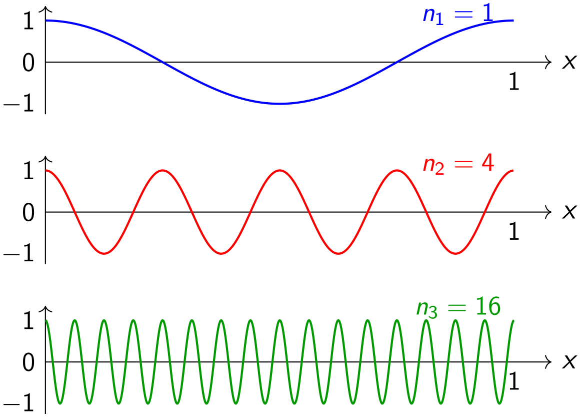
Hadamard gap condition

$$n_{k+1} \geq q\, n_k, \quad q > 1, \ \ \forall\, k \geq 1,$$

- "Lacunary" means "**with gaps**"
- Each frequency is at least q times the previous one
- The frequencies grow **at least geometrically**
- the gaps between frequencies increase rapidly
- The coefficients c_k determine the amplitude

Example

$\cos(2\pi n_k x)$, $\quad n_k = 4^{k-1} \quad$ (q=4), $\quad n_1 = 1,\ n_2 = 4,\ n_3 = 16$, $\quad$ frequencies: $\ 1, 4, 16, 64, \ldots$



The frequency determines the number of oscillations on $[0, 1]$.

What makes lacunary sums special?

- Different frequencies oscillate at very different speeds
- They do not interact strongly
- Lacunary sums behave similarly to sums of almost independent random variables

$$\text{Deterministic structure} \quad \Rightarrow \quad \text{probabilistic behavior}$$

Key message

Lacunary sums bridge harmonic analysis and probability

Orthonormality and variance

Consider the functions

$$\phi_k(x) = \sqrt{2}\cos(2\pi n_k x).$$

Then

$$f_N(x) = \sum_{k=1}^{N} c_k \phi_k(x)$$

The functions ϕ_k are orthonormal, so that

$$\int_0^1 \phi_k(x)\phi_j(x)\,dx = \delta_{kj} = \begin{cases} 0, & k \neq j, \\ 1, & k = j \end{cases}$$

Expectation: $\mathbb{E}[f_N] = \int_0^1 f_N(x)\,dx = 0$

Variance: $\sigma^2(N) = \mathrm{Var}(f_N) = \mathbb{E}[f_N^2] - (\mathbb{E}[f_N])^2 = \sum_{k=1}^{N} c_k^2$

Normalization

We study the **normalized sum**

$$\nu_N(x) = \frac{f_N(x)}{\sigma(N)}, \qquad \sigma^2(N) = \sum_{k=1}^{N} c_k^2.$$

Then

$$\mathbb{E}[\nu_N] = 0, \qquad \mathrm{Var}(\nu_N) = 1.$$

Assumption

$$\sum_{k=1}^{\infty} c_k^2 = \infty$$

Hence

$$\sigma^2(N) \to \infty,$$

so the normalized sums exhibit non-trivial behavior.

Classical asymptotic results and motivation

Lacunary trigonometric sums are deterministic, but they behave similarly to sums of almost independent random variables.

Classical **asymptotic results** describe the behavior as $N \to \infty$:

- Central Limit Theorem (CLT) $\Rightarrow$ Gaussian limit;
- Law of the Iterated Logarithm (LIL) $\Rightarrow$ maximal fluctuations;
- Large Deviation Principle (LDP) $\Rightarrow$ exponential estimates for rare events

Our focus: explicit tail bounds for fixed finite N

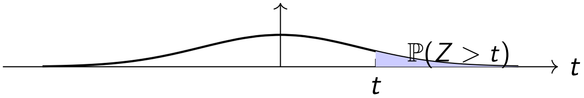
Motivation: applications to Fourier analysis, signal processing, and stochastic processes.

Gaussian limit and tail

Central Limit Theorem for lacunary sums

$$\nu_N(x) \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as } N \to \infty.$$

$$\mathbb{P}(\nu_N > t) \approx \mathbb{P}(Z > t), \qquad Z \sim \mathcal{N}(0,1).$$



$$\mathbb{P}(Z > t) = 1 - \Phi(t), \qquad \Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

The main question

For $t > 0$, define the **upper tail**

$$T[\nu_N](t) = \mathbb{P}\{\nu_N > t\} = \mathbb{P}\{x \in [0,1] : \nu_N(x) > t\}$$

Our goal

To derive non-asymptotic bounds for the tail probabilities.

More precisely, **for fixed N**, we provide explicit exponential estimates for the tail probabilities in terms of the lacunarity parameter q and the normalizing factor $\sigma(N)$.

The bounds show two regimes:

- subgaussian behavior for small deviations;
- stretched-exponential behavior for larger deviations.

The key issue is to understand the transition between the two regimes.

Moment generating function and cumulants

Let $\lambda \in \mathbb{R}$. For the normalized lacunary sum ν_N, define

Moment generating function (MGF):

$$M_{\nu_N}(\lambda) = \mathbb{E}[e^{\lambda \nu_N}]$$

Cumulant generating function (CGF):

$$K_{\nu_N}(\lambda) = \log M_{\nu_N}(\lambda)$$

Near the origin,

$$K_{\nu_N}(\lambda) = \sum_{r=1}^{\infty} \frac{\kappa_r(\nu_N)}{r!} \lambda^r,$$

where $\kappa_r(\nu_N)$ denotes the r-th **cumulant**, i.e. the r-th derivative of K_{ν_N} at $\lambda = 0$:

$$\kappa_r(\nu_N) = K_{\nu_N}^{(r)}(0).$$

Cumulants generalize the notions of mean and variance.

In particular:

$$\kappa_1(\nu_N) = \mathbb{E}[\nu_N], \qquad \kappa_2(\nu_N) = \mathrm{Var}(\nu_N).$$

Since the normalized lacunary sums ν_N have zero mean and unit variance, we have

$$\kappa_1(\nu_N) = 0, \qquad \kappa_2(\nu_N) = 1.$$

Therefore,

$$K_{\nu_N}(\lambda) = \frac{\lambda^2}{2} + \frac{\kappa_3(\nu_N)}{3!}\lambda^3 + \sum_{r\geq 4} \frac{\kappa_r(\nu_N)}{r!}\lambda^r$$

- For a Gaussian random variable, all cumulants of order $r \geq 3$ vanish.
- For lacunary sums higher cumulants do not vanish exactly.
- The cubic term is the first non-Gaussian correction.
- It measures the residual dependence among the lacunary frequencies.

Third cumulant estimate

The third cumulant measures the first non-Gaussian correction.

The lacunarity condition allows quantitative control of the third cumulant

$$\boxed{|\kappa_3(\nu_N)| \le \widetilde{\kappa}_N}$$

where

$$\widetilde{\kappa}_N = \frac{2\sqrt{2q}}{q-1}\frac{1}{\sigma(N)}$$

Thus the third cumulant is small, but generally non-zero.

Local MGF bound

This leads to the local MGF estimate, for $|\lambda|$ sufficiently small:

$$\boxed{\mathbb{E}[e^{\lambda \nu_N}] \leq \exp\left(\frac{\lambda^2}{2} + \frac{\widetilde{\kappa}_N}{3}|\lambda|^3\right)}$$

where

$$\widetilde{\kappa}_N = \frac{2\sqrt{2q}}{q-1}\frac{1}{\sigma(N)}$$

- the quadratic term is the Gaussian contribution;
- the cubic term is the first non-Gaussian correction.

Chernoff bound

Since

$$\mathbb{P}(\nu_N \geq t) = \mathbb{P}(e^{\lambda \nu_N} \geq e^{\lambda t}),$$

applying Markov's inequality to $X = e^{\lambda \nu_N}$ we obtain

$$\mathbb{P}(e^{\lambda \nu_N} \geq e^{\lambda t}) \leq \frac{\mathbb{E}[e^{\lambda \nu_N}]}{e^{\lambda t}}$$

and therefore

$$T[\nu_N](t) = \mathbb{P}(\nu_N \geq t) \leq e^{-\lambda t}\mathbb{E}[e^{\lambda \nu_N}].$$

Using the MGF bound, for $\lambda > 0$ sufficiently small,

$$\boxed{T[\nu_N](t) \leq \exp\left(-\lambda t + \frac{\lambda^2}{2} + \frac{\widetilde{\kappa}_N}{3}\lambda^3\right)}$$

We then optimize the right-hand side with respect to λ.

Optimal parameter

The exponent to maximize in the dual form is

$$\lambda t - \frac{\lambda^2}{2} - \frac{\widetilde{\kappa}_N}{3}\lambda^3.$$

The optimal value satisfies

$$t = \lambda + \widetilde{\kappa}_N \lambda^2.$$

Hence

$$\widehat{\lambda}(t) = \frac{-1 + \sqrt{1 + 4\widetilde{\kappa}_N t}}{2\widetilde{\kappa}_N}.$$

This is the Legendre–Fenchel optimization behind the tail bound.

Optimized upper bound

If $\widehat{\lambda}(t)$ belongs to the admissible MGF range, then

$$\boxed{T[\nu_N](t) \leq \exp\left(-\frac{\widehat{\lambda}(t)^2}{2} - \frac{2}{3}\widetilde{\kappa}_N \widehat{\lambda}(t)^3\right)}$$

This formula is explicit and depends on:

- the deviation level t;
- the variance scale $\sigma(N)$;
- the lacunarity parameter q.

Two deviation regimes

Small deviations: subgaussian behavior

When $t \ll \sigma(N)$,

$$T[\nu_N](t) \lesssim \exp\left(-\frac{t^2}{2}\right)$$

Moderate/Large deviations: stretched-exponential regime

When $t \gg \sigma(N)$,

$$T[\nu_N](t) \lesssim \exp\left(-c(q)\,\sigma(N)^{1/2}t^{3/2}\right)$$

subgaussian behavior at small scale; stretched-exponential behavior at larger scale

Small deviations: subgaussian regime - Sketch of the proof

The optimized bound implies the weaker estimate

$$T[\nu_N](t) \leq \exp\left(-\frac{\widehat{\lambda}(t)^2}{2}\right).$$

The optimal parameter $\widehat{\lambda}(t)$ satisfies

$$t = \lambda + \widetilde{\kappa}_N \lambda^2, \qquad \widetilde{\kappa}_N = \frac{2\sqrt{2q}}{q-1}\frac{1}{\sigma(N)}.$$

Using an elementary inequality for the square root, we obtain

$$t - \widetilde{\kappa}_N t^2 \leq \widehat{\lambda}(t) < t.$$

Hence

$$\text{for} \quad t \ll \sigma(N) \quad \Rightarrow \quad \widehat{\lambda}(t) \approx t \quad \Rightarrow \quad T[\nu_N](t) \lesssim \exp\left(-\frac{t^2}{2}\right)$$

Moderate/Large deviations: stretched-exponential regime

The optimality condition is

$$t = \lambda + \widetilde{\kappa}_N \lambda^2.$$

For $t \gg \sigma(N)$, the quadratic term in λ becomes relevant:

$$t \approx \widetilde{\kappa}_N \lambda^2.$$

Since

$$\widetilde{\kappa}_N = \frac{2\sqrt{2q}}{q-1} \frac{1}{\sigma(N)},$$

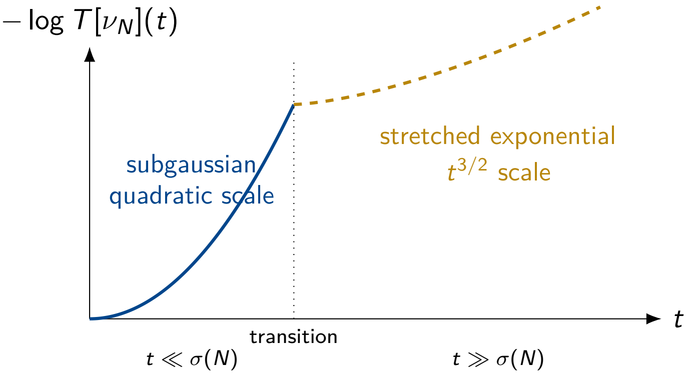
we get

$$\widehat{\lambda}(t) \approx \sqrt{\sigma(N)t}$$

Substitution gives

$$T[\nu_N](t) \lesssim \exp\left(-c(q)\,\sigma(N)^{1/2} t^{3/2}\right).$$

Qualitative picture: plotting the scale of the exponent governing the tail probability



- For $t \ll \sigma(N)$, the tail is essentially subgaussian.
- For $t \gg \sigma(N)$, the cubic correction produces a stretched-exponential decay.
- The transition is gradual and occurs around $t \sim \sigma(N)$, i.e. it is governed by $t/\sigma(N)$.

Example 1: bounded coefficients

Let

$$c_k = 1, \qquad n_k = 2^k.$$

Then

$$q = 2, \qquad \sigma^2(N) = N, \qquad \sigma(N) = \sqrt{N}.$$

Hence

$$\widetilde{\kappa}_N = \frac{2\sqrt{2q}}{q-1}\frac{1}{\sigma(N)} = \frac{4}{\sqrt{N}}.$$

- For small deviations: $0 \leq t \leq \sqrt{N}$

$$T[\nu_N](t) \lesssim e^{-ct^2}$$

- For larger deviations: $t \geq \sqrt{N}$

$$T[\nu_N](t) \lesssim \exp\left(-cN^{1/4}t^{3/2}\right)$$

Example 2: slowly decaying coefficients

Let

$$c_k = k^{-1/2}, \qquad n_k = 3^k.$$

Then

$$q = 3, \qquad \sigma^2(N) = \sum_{k=1}^{N} \frac{1}{k} \sim \log N, \qquad \sigma(N) \sim (\log N)^{1/2}, \qquad \widetilde{\kappa}_N \asymp (\log N)^{-1/2}.$$

- For small deviations: $0 \le t \ll (\log N)^{1/2}$, the tail remains subgaussian

$$T[\nu_N](t) \lesssim e^{-ct^2}.$$

- For larger deviations: $t \gg (\log N)^{1/2}$, the stretched-exponential estimate is

$$T[\nu_N](t) \lesssim \exp\left(-c(\log N)^{1/4} t^{3/2}\right).$$

Summary and possible directions

Summary

- Lacunary sums behave almost like independent sums.
- The Gaussian term gives the usual t^2 exponent.
- Arithmetic resonances create a cubic cumulant correction.
- Optimizing the Chernoff bound transforms this cubic term into a $t^{3/2}$ exponent.

Possible directions

- obtaining matching lower tail estimates;
- allowing random coefficients;
- extending the theory to vector-valued lacunary sums.

Some references

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Thank you!