On explicit Reeb spaces of real analytic functions which are not homeomorphic to any finite graph

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- The speaker now works at Kyushu-Sangyo University Kisokyoiku-Support-Center (education), mainly. This study is independent of this.

Main theme: Reeb spaces of real analytic (algebraic) functions.

- The Reeb space R_c of a continuous real-valued function $c : X \to \mathbb{R}$: the set R_c of all connected components of level sets of c topologized with the natural quotient topology $X/\sim c$.
  - Important in geometry of manifolds since around 1950.
  - A 1-dim. metrizable Peano continuum for a metrizable Peano continuum X: a Peano continuum is compact, connected and locally-connected Hausdorff space (Gelbukh 2023).
- The Reeb Graph R_c: the graph whose vertex set consists of all points corresponding to connected components containing some critical points of c (: X is a smooth manifold with no boundary and c is also smooth).

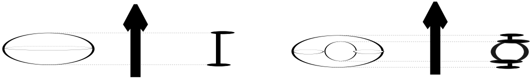

Figure: The Reeb graph of a natural height function of a sphere of dim. ≥ 2. The Reeb graph of a natural function on a torus $S^1 \times S^1 \subset \mathbb{R}^3$ in the 3-dim. Euclidean space $\mathbb{R}^3$.

Today we mainly present.

- N. Kitazawa, *Real algebraic functions on closed manifolds whose Reeb graphs are given graphs*, Methods of Functional Analysis and Topology Vol. 28 No. 4 (2022), 302–308, 2023 (Related to our proof of Theorem 1 and Theorem 3).
- N. Kitazawa, *Some remarks on real algebraic maps which are topologically special generic maps*, arXiv:2312.10646, 2025 (Related to our proof of Theorems 1 and 3).
- N. Kitazawa, *Reconstructing real algebraic maps locally like moment-maps with prescribed images and compositions with the canonical projections to the* 1*-dimensional real affine space*, the title has changed from previous versions, arXiv:2303.10723, 2024 (Related to our proof of Theorem 2).
- N. Kitazawa, *A note on Reeb spaces of some explicit real analytic functions*, submitted to a refereed journal, arXiv:2601.11648, 2026 (Theorems 1 and 2).

Reeb (di)graphs of smooth functions on manifolds X.

Definition 1

The Reeb space R_c of a (continuous) function $c : X \to \mathbb{R}$ on a topological space X is defined as follows.

- $R_c := X/\sim c$ is defined by the equivalence relation $\sim c$ on X: $p_1 \sim_c p_2 \leftrightarrow p_1$ and p_2 are in a same component of a same level set $c^{-1}(q)$. Let $q_c : X \to R_c$ denote the quotient space.
- Let X be a smooth manifold of dimension $m \geq 2$ with no boundary and $c : X \to \mathbb{R}$ a smooth function.

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By the unique continuous function $\bar{c}$ with $c = \bar{c} \circ q_c$, $(R_c, \bar{c})$ is a so-called digraph (the Reeb digraph) s.t. the orientation is induced naturally by $\bar{c}$ (if the Reeb graph is obtained).

Additional exposition on Reeb (di)graphs.

Problem 1 (The differentiable (smooth) case: Sharko (2006). The real algebraic case: K (2023–).)

Can we reconstruct nice smooth (real algebraic) functions with prescribed finite Reeb graphs?

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  $\Rightarrow$ We cannot use bump functions, partitions of the unity $\cdots$.

Fundamental terminologies, notions and notation on smooth manifolds.

- $\mathbb{R}^k$: the k-dim. Euclidean space ($\mathbb{R}^1 := \mathbb{R}$).
  $\Rightarrow$ A simplest k-dim. smooth manifold and a Riemannian manifold equipped with the standard Euclidean metric. It is also a non-singular real algebraic manifold: the k-dim. real affine space.
- $||p||$: the distance between $p \in \mathbb{R}^k$ and the origin $0 \in \mathbb{R}^k$.
- S^k (D^{k+1}):=$\{p \in \mathbb{R}^{k+1} \mid ||x|| =(\text{resp. } \leq)1\}$: the k-dim. unit sphere (resp. the $(k+1)$-dim. unit disk).
- X^l : l-dim. smooth manifold X ("l" in "X^l" is its dimension).

Fundamental terminologies, notions and notation on smooth maps.

- $\pi_{k,k'} : \mathbb{R}^k \to \mathbb{R}^{k'}$ $(k > k' \geq 1)$ is defined by $\pi_{k,k'}(x_1, x_2) := x_1$ $((x_1, x_2) \in \mathbb{R}^k = \mathbb{R}^{k'} \times \mathbb{R}^{k-k'})(\mathbb{R}^{k'} \times \mathbb{R}^0 = \mathbb{R}^{k'})$: $\pi_{k,k'}|_{S^{k-1}}$ is the canonical projection of the unit sphere S^{k-1}).
- $f : M^m \to N^n$: a smooth map between smooth manifolds.
  $p \in M^m$ is a singular point of f (for $n = 1$ it is also called a critical point) : at p the rank of the differential $df_p < \min\{m, n\}$ and $f(p)$ is a singular value of f (: in $n = 1$ it is also called a critical value).

Other fundamental notation, rules etc.

$\pi_{k_1+k_2,k_2} : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_1}$ s.t. $k_1, k_2 > 0$ and $\pi_{k_1+k_2,k_2}(x_1, x_2) := x_1$.

- Real algebraic (analytic) manifolds: unions of some connected components of the zero sets of real polynomial maps (resp. analytic maps into $\mathbb{R}^n$) and non-singular: non-singular ones are defined naturally via implicit function theorem (e.g. the unit sphere $S^k \subset \mathbb{R}^{k+1}$ is of simplest ones).
- Real algebraic (analytic) maps here: the compositions of natural embeddings of real algebraic (resp. analytic) manifolds $M \subset \mathbb{R}^{k_1+k_2}$ with the canonical projection $\pi_{k_1+k_2,k_1} : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_1}$ (e.g. the canonical projection $\pi_{k_1+k_2,k_1}|_{S^{k_1+k_2-1}}$ is of simplest ones).
  $\to$ Real algebraic maps are of a certain class of specific entire rational maps !

Reeb spaces of smooth functions which are not homeomorphic to any finite graph?

Problem 2

Can we have Reeb spaces of smooth functions which are not homeomorphic to any finite graph?

Gelbukh, Saeki etc. have presented examples by using bump functions or the projections of open sets in Euclidean spaces.

Problem 3 (2026 –K)

Can we have such cases by real analytic functions or smooth functions which are densely real analytic?

One of our related new result.

Theorem 1 (2026– K)

$\exists$ G_1 : an infinite connected graph, $\forall m \geq 2$: an integer, $\exists$ an m-dimensional real analytic manifold $X_m \subset \mathbb{R}^{m+1}$ represented as $X_m = {e_m}^{-1}(0)$ for some real analytic and elementary function $e_m : \mathbb{R}^{m+1} \to \mathbb{R}$, and the following hold.

1. *$R_{\pi_{m+1,1}|_{X_m}}$ is regarded as the Reeb graph and isomorphic to G_1.*
2. *The quotient map $q_{\pi_{m+1,1}|_{X_m}}$ is proper and the preimage of each point having no critical point of the function $\pi_{m+1,1}|_{X_m}$ is diffeomorphic to S^{m-1}.*

Another related new result.

Theorem 2 (2026– K)

$\exists$ G_2 : a metrizable Peano continuum which is not homeomorphic to any graph, $\forall (m_1, m_2)$: a pair of integers $m_1, m_2 \geq 2$, $\exists$ an $(m_1 + m_2)$-dimensional smooth compact submanifold $X_{m_1,m_2} \subset \mathbb{R}^{m_1+m_2+2}$ with no boundary and represented as $X_{m_1,m_2} := {e_{m_1,m_2}}^{-1}(0)$ for some smooth map $e_{m_1,m_2} : \mathbb{R}^{m_1+m_2+2} \to \mathbb{R}^2$ which is real analytic and each component of which is an elementary function outside some subset $Z_{m_1,m_2} \subset \mathbb{R}^{m_1+m_2+2}$ of Lebesgue measure 0, and the following hold.

3. *$\exists p_{G_2}$, $G_2 - \{p_{G_2}\}$ is homeomorphic to a graph, the preimage ${q_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}}}}^{-1}(p_{G_2})$ is $\{0\} \subset \mathbb{R}^{m_1+m_2+2}$, and $R_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}-\{0\}}}$ is regarded as the Reeb graph of $\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}-\{0\}}$.*
4. *$R_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}}}$ is homeomorphic to G_2.*

Some of idea for our proof of Theorem 1.

Consider $D_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{\sin^2(x_2)}{2({x_2}^2+1)} \leq x_1 \leq \frac{1}{{x_2}^2+1}\}$.
We can have $X_m := \{(x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^{m-1} \mid e_m(x_1, x_2, y) = (x_1 - \frac{\sin^2(x_2)}{2({x_2}^2+1)})(\frac{1}{{x_2}^2+1} - x_1) - ||y||^2 = 0\}$ and $\pi_{m+1,2}(X_m) = D_1$.

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This is of a generalized case of $D_0 := D^n$ (the unit disk in $\mathbb{R}^n$ and we have $X_m := S^m$ for $m \geq n$: 2023– K)

(2023– K) We have considered a connected region $D \subset \mathbb{R}^n$ surrounded by the disjoint union $\sqcup_{j=1}^{l} S_j$ of some unions S_j of connected components of the zero sets of real polynomials $\{f_j\}_{j=1}^{l}$ which are real algebraic s.t.
$\exists$ (connected open neighborhood $\exists U_D$ of the closure $\overline{D}$) and

- $U_D \bigcap D = (\bigcap_{j=1}^{l} \{x \mid f_j(x) > 0\}) \bigcap D$.
- $S_j = U_D \bigcap \{x \mid f_j(x) = 0\}$.

$X_m := \{(x, y) \in D \times \mathbb{R}^{m-n+1} \mid \prod_{j=1}^{l}(f_j(x)) - {||y||}^2 = 0\}$.

A topic related to Theorem 1.

Definition 2 (Bodin, Popescu-Pampu, Sorea ($n = 2$: 2022–), Savi(n is general: 2026))

A pair (G, c_G) as follows is a graph with a good $\mathbb{R}^n$-orientation.

- A finite graph G is embedded in $\mathbb{R}^n$ via $c_G : G \to \mathbb{R}^n$.
- The restriction $\pi_{n,1} \circ c_G|_e$ to each edge e is injective and the restriction to the vertex set is injective: a finite graph G is oriented naturally.
- The degree of each vertex of G is 1 or 3. A vertex where $\pi_{n,1} \circ c_G$ has a local extremum is of degree 1.

Theorem 3 (2023 –K)

A graph (G, c_G) with a good $\mathbb{R}^n$-orientation is isomorphic to the Reeb digraph of the real algebraic function $\pi_{m,n}|_{X_m}$ on some $m \geq n$ dimensional real agebraic manifold $X_m \subset \mathbb{R}^{m+1}$ $(\forall m \geq n)$.

Some remark on Theorem 3.

- For Theorem 3, obtaining regions naturally collapsing to the graphs by approximation is important: due to Bodin, Popescu-Pampu & Sorea (2022) and Savi (2026). We apply our construction of real algebraic manifolds and maps.
- The speaker has pointed out that X_m can be the zero set of some real polynomial (2023– K).
- The speaker has also pointed out that "$\prod_{j=1}^{l}(f_j(x)) - ||y||^2 = 0$" can be generalized to some generalize forms (2023–K).
- The speaker has also obtained cases where graphs are not of the type in Definition 2.

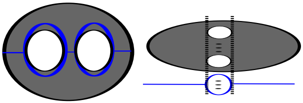


Figure: Regions surrounded by circles of fixed radii and graphs they collapse to ($n = 2$): the figure in the left (right) is (resp. is not) for Definition 2 or Theorem 3.

Some of idea for our proof of Theorem 2.

Let $s(x) := \begin{cases} e^{-x^2}\sin^2(-\frac{1}{x^2}) \ (x \neq 0) \\ 0 \ (x = 0) \end{cases}$

Consider $D_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid r - \frac{(x_1-p_1)^2}{a_1} - \frac{(x_2-p_2)^2}{a_2} \geq 0\} \bigcap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - s(x_2) \geq 0\} \ni 0$ with $a_1, a_2, p_1, p_2, r > 0$ and a_1 being sufficiently large.

We can have

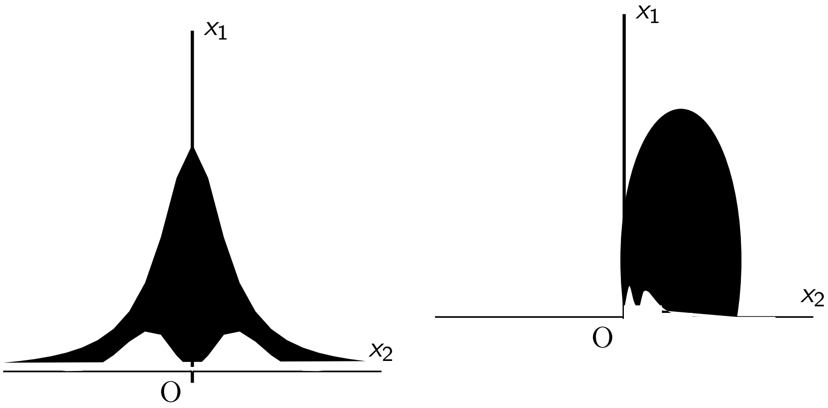
$X_{m_1,m_2} := \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^{m_1+m_2} \mid e_{m_1,m_2}(x_1, x_2, y_1, y_2) := (r - \frac{(x_1-p_1)^2}{a_1} - \frac{(x_2-p_2)^2}{a_2} - ||y_1||^2, x_1 - s(x_2) - ||y_2||^2) = 0 \in \mathbb{R}^2\}$

and $\pi_{m+1,2}(X_{m_1,m_2}) = D_2$.

Related to Theorems 2 and 3 and related presented arguments, he has also considered regions surrounded by curves of degree 1 or 2 intersecting in the transversal way (2023–K) etc..

Some additional comments.

$\pi_{m+1,2}(X_m)$ and $\pi_{m_1+m_2+2,2}(X_{m_1,m_2})$ for Theorems 1–2.



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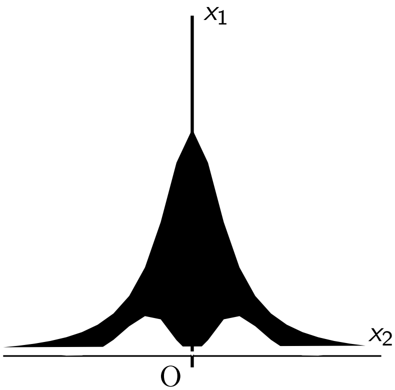

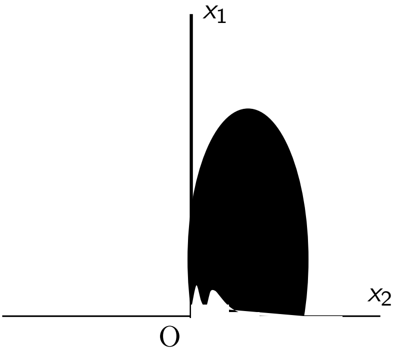

- By composing $\pi_{2,1}$, our desired functions are obtained.
- To see that the Reeb spaces are infinite graphs or Peano continua, apply general theory from Gelbukh and Saeki (in 2020s).

References.

A. Bodin, P. Popescu-Pampu and M. S. Sorea, *Poincaré-Reeb graphs of real algebraic domains*, Revista Matemática Complutense, https://link.springer.com/article/10.1007/s13163-023-00469-y, 2023.

I. Gelbukh, *On the topology of the Reeb graph*, Publicationes Mathematicae Debrecen 104(3–4) (2023), 343–365.

N. Kitazawa, *On Reeb graphs induced from smooth functions on* 3*-dimensional closed orientable manifolds with finitely many singular values*, Topol. Methods in Nonlinear Anal. Vol. 59 No. 2B, 897–912.

N. Kitazawa, *On Reeb graphs induced from smooth functions on closed or open surfaces*, Methods of Functional Analysis and Topology Vol. 28 No. 2 (2022), 127–143, arXiv:1908.04340.

N. Kitazawa, *Constructing Morse functions with given Reeb graphs and level sets*, accepted for publication in Topological Methods in Nonlinear Analysis, arXiv:2108.06913 (, where the title has been changed from the title there), 2025.

Y. Masumoto and O. Saeki, *A smooth function on a manifold with given Reeb graph*, Kyushu J. Math. 65 (2011), 75–84.

L. P. Michalak, *Realization of a graph as the Reeb graph of a Morse function on a manifold*. Topol. Methods in Nonlinear Anal. 52 (2) (2018), 749–762, arXiv:1805.06727.

G. Reeb, *Sur les points singuliers d´une forme de Pfaff complétement intègrable ou d´une fonction numérique*, Comptes Rendus Hebdomadaires des Séances de I´Académie des Sciences 222 (1946), 847–849.

O. Saeki, *Reeb spaces of smooth functions on manifolds*, International Mathematics Research Notices, maa301, Volume 2022, Issue 11, June 2022, 8740–8768, arXiv:2006.01689.

E. Savi, *Algebraic realization of stable Poincaré-Reeb graphs*, arXiv:2602.18780.

V. Sharko, *About Kronrod-Reeb graph of a function on a manifold*, Methods of Functional Analysis and Topology 12 (2006), 389–396.

Thank you !