Local behavior of ring Q-homeomorphisms with respect to the p-modulus under exponential integrability of the function Q

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p-module of a family of curves

Let Γ be a family of curves γ in the complex plane $\mathbb{C}$. A Borel function $\rho : \mathbb{C} \to [0, \infty]$ is called *admissible* for Γ, denoted $\rho \in \mathrm{adm}\,\Gamma$, if

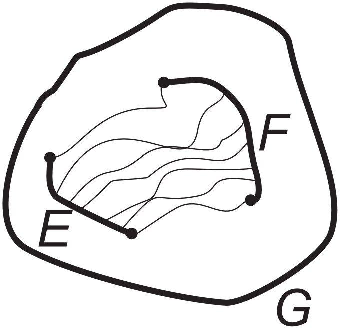
(1) $$\int_\gamma \rho(\mathrm{z})|\mathrm{dz}| \geqslant 1$$

for each (locally rectifiable) curve $\gamma \in \Gamma$.
Let $\mathrm{p} > 1$. The p-*module* of the family Γ is defined as the quantity

(2) $$\mathrm{M_p}(\Gamma) = \inf_{\rho \in \mathrm{adm}\,\Gamma} \int_{\mathbb{C}} \rho^{\mathrm{p}}(\mathrm{z})\,\mathrm{dxdy}\,.$$

p-module of a family of curves

For arbitrary sets E, F, and G in $\mathbb{C}$, we denote by $\Delta(E,F;G)$ the family of all curves $\gamma : [a,b] \to \mathbb{C}$ that join E and F in G, that is, such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for all $a < t < b$.



Condenser

A *condenser* is a pair $\mathscr{E} = (\mathrm{A}, \mathrm{C})$, where $\mathrm{A} \subset \mathbb{C}$ is open and C is a nonempty compact set contained in A.
A condenser $\mathscr{E}$ is a *ringlike condenser* if $\mathrm{A} \setminus \mathrm{C}$ is a ring, i.e., $\mathrm{A} \setminus \mathrm{C}$ is a domain whose complement $\overline{\mathbb{C}} \setminus (\mathrm{A} \setminus \mathrm{C})$ has exactly two components, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
A condenser $\mathscr{E} = (\mathrm{A}, \mathrm{C})$ is said to *be in a domain* G if $\mathrm{A} \subset \mathrm{G}$.

p-capacity of the condenser

Let $\mathscr{E} = (A, C)$ be a condenser. Denote by $\mathscr{C}_0(A)$ a set of continuous functions $u : A \to \mathbb{R}^1$ with compact support. Let $W_0(\mathscr{E}) = W_0(A, C)$ be a family of nonnegative functions $u : A \to \mathbb{R}^1$ such that 1) $u \in \mathscr{C}_0(A)$, 2) $u(z) \geqslant 1$ for $z \in C$, 3) u belongs to the class ACL and

(3) $$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}.$$

For $p \geqslant 1$, the quantity

(4) $$\operatorname{cap}_p \mathscr{E} = \operatorname{cap}_p (A, C) = \inf_{u \in W_0(\mathscr{E})} \int_A |\nabla u|^p \, dxdy$$

is called p-*capacity* of the condenser $\mathscr{E}$.

Q-homeomorphisms

Let G be a domain in $\mathbb{C}$ and let $Q : G \to [0, \infty]$ be a measurable function. A homeomorphism $f : G \to \mathbb{C}$ is called a Q-*homeomorphism with respect to* p-*modulus* if

(5) $$M_p(f\Gamma) \leqslant \int\limits_G Q(z)\,\rho^p(z)\,dm(z)$$

for every path family Γ in G and every function $\rho \in \operatorname{adm}\Gamma$.
The notion of Q-homeomorphisms with respect to the p-modulus for $p = n$ was first introduced in the papers [1], [2].

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[1] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *Mappings with finite length distortion*, J. Anal. Math., **93** (2004), 215–236.

[2] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *On Q-homeomorphisms*, Ann. Acad. Sci. Fenn. Ser. A1. Math., **30** (2005), 49–69.

Ring Q-homeomorphisms

The following notion in a natural way localizes and generalizes the notion of a Q-homeomorphism with respect to the p-modulus.

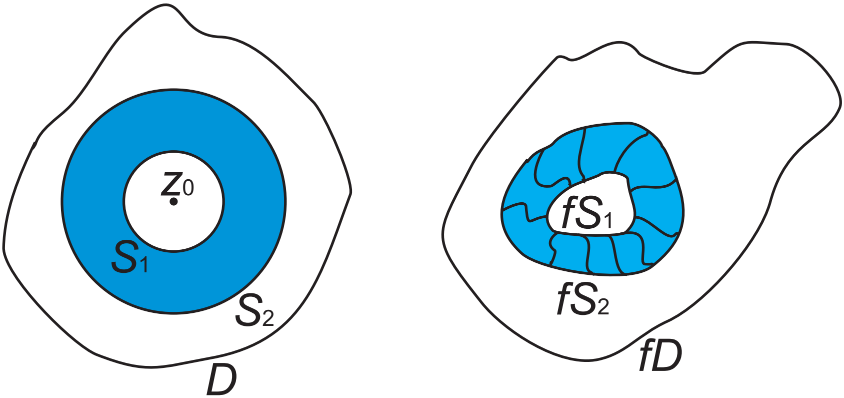
Let D be a domain in the complex plane $\mathbb{C}$ and let $Q : D \to [0, \infty]$ be a Lebesgue measurable function. A homeomorphism $f : D \to \mathbb{C}$ is called a *ring* Q*-homeomorphism with respect to the* p*-modulus* at a point $z_0 \in D$ if the inequality

(6) $$M_p(\Delta(fS_1, fS_2; f\mathbb{A})) \leqslant \int\limits_{\mathbb{A}} Q(z)\, \eta^p(|z - z_0|)\, dxdy$$

holds for every ring $\mathbb{A} = \mathbb{A}(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and the circles $S_i = S(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}$, $i = 1, 2$, where $0 < r_1 < r_2 < d_0$, $d_0 = \mathrm{dist}(z_0, \partial D)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1.$$

Ring Q-homeomorphisms



Ring Q-homeomorphisms

Ring Q-homeomorphisms with respect to the p-module in the space $\mathbb{R}^n$, $n \geqslant 2$, were studied in the following works:

- for $p = n$ in [3], [4], [5], [6], [7];

[3] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2009). *Moduli in Modern Mapping Theory.* New York: Springer Science + Business Media, LLC.

[4] Ryazanov, V., Srebro, U., Yakubov, E. (2003). *Degenerate Beltrami equation and radial Q-homeomorphisms.* Preprint, Department of Mathematics, University of Helsinki, 369.

[5] Ryazanov, V., Srebro, U., Yakubov, E. (2007). The Beltrami equation and ring homeomorphisms. *Ukrain. Math. Bull., 4*(1), 79–115.

[6] Ryazanov, V.I., Sevost'yanov, E.A. (2008). Toward the theory of ring Q-homeomorphisms. *Israel J. Math., 168*, 101–118.

[7] Salimov, R.R., Smolovaya, E.S. (2010). On the order of growth of ring Q-homeomorphisms at infinity. *Ukrainian Mathematical Journal, 62*(6), 961–969.

Ring Q-homeomorphisms

- for $1 < p < n$ in [8], [9], [10], [11];

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[8] Golberg, A., Salimov, R. (2014). Logarithmic Holder continuity of ring homeomorphisms with controlled p-module. *Complex Variables and Elliptic Equations*, *59*(1), 91–98.

[9] Golberg, A., Salimov, R., Sevost'yanov, E. (2014). Distortion estimates under mappings with controlled p-module. *Ann. Univ. Bucharest, Ser. Math.*, *5 (LXIII)*, 95–114.

[10] Salimov, R.R. (2013). One property of ring Q-homeomorphisms with respect to a p-module. *Ukrainian Mathematical Journal*, *65*(5), 806–813.

[11] Salimov, R.R. (2014). To a theory of ring Q-homeomorphisms with respect to a p-modulus. *Journal of Mathematical Sciences*, *196*, 679–692.

Ring Q-homeomorphisms

- for $p > n$ in [12], [13], [14];
- in the complex plane for $p > 2$ in [15], [16], [17], [18].

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[12] Klishchuk, B., Salimov, R., Stefanchuk, M. (2025). On the asymptotic behavior of the diameter of the image of a ball at infinity. *Ukr. Math. J.*

[13] Salimov, R., Klishchuk, B. (2023). On the behavior of one class of homeomorphisms at infinity. *Ukrainian Mathematical Journal, 74*, 1617–1628.

[14] Klishchuk, B., Salimov, R., Stefanchuk, M. (2023). On the asymptotic behavior at infinity of one mapping class. *Proceedings of the International Geometry Center, 16*(1), 50–58.

[15] Salimov, R., Klishchuk, B. (2018). Extremal problem for the area of the image of a disk. *Journal of Mathematical Sciences, 234*(3), 373–380.

[16] Petkov, I.V., Salimov, R.R., Stefanchuk, M.V. (2023). On the distortion of the disk image diameter. *J. Math. Sci., 274*, 352–369.

[17] Stefanchuk, M.V. (2024). On exponential asymptotics of one class of homeomorphisms at a point of the complex plane. *Proceedings of the International Geometry Center, 17*(2), 158–170.

[18] Stefanchuk, M.V. (2024). On exponential asymptotics of ring Q-homeomorphisms at infinity. *J. Math. Sci., 282*(1), 83–92.

In the complex plane, the theory of ring Q-homeomorphisms with respect to the p-module is used in the study of the asymptotic properties of regular solutions of nonlinear Beltrami equations, see [19], [20], [21].

[19] Golberg, A., Salimov, R. (2019). Nonlinear Beltrami equation. *Complex Variables and Elliptic Equations*, *65*(1), 6–21.

[20] Salimov, R., Stefanchuk, M. (2024). Finite Lipschitzness of regular solutions to nonlinear Beltrami equation. *Complex Variables and Elliptic Equations*, *69*(6), 913–923.

[21] Petkov, I., Salimov, R., Stefanchuk, M. (2025). On Local Hölder and Lipschitz Continuities of Solutions to Nonlinear Beltrami Equation. *New Tools in Mathematical Analysis and Applications. Trends in Mathematics*, 109–119.

We denote $B_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$.

Lemma 1. *Let* $f : \mathbb{B} \to \mathbb{C}$ *be a ring* Q*-homeomorphism with respect to the* p*-modulus at a point* $z_0 = 0$, $p > 1$. *Suppose, for some numbers* $\lambda > 0$ *and* $r_0 \in (0,1)$, *the following condition*

$$Q_0 = \int\limits_{B_{r_0}} \exp(\lambda Q(z))\, dm(z) < \infty \tag{7}$$

holds. Then

$$\mathrm{cap}_p\left(fB_{r_2}, f\overline{B}_{r_1}\right) \leqslant \frac{\pi}{\lambda} \cdot \frac{r_2^2}{(r_2 - r_1)^p} \cdot \log \frac{Q_0}{\pi r_2^2} \tag{8}$$

for every r_1, r_2 *such that* $0 < r_1 < r_2 < r_0$.

Estimate for the area of the image of a disc

Theorem 1. *Let* $f:\mathbb{B}\to\mathbb{C}$ *be a ring* Q*-homeomorphism with respect to the* p*-modulus at a point* $z_0=0$, $1<p<2$. *Suppose, for some numbers* $\lambda>0$ *and* $r_0\in(0,1)$, *the condition*

$$Q_0=\int\limits_{B_{r_0}}\exp\left(\lambda Q(z)\right)dm(z)<\infty \tag{9}$$

is fulfilled. Then, for all $r\in\left(0,\frac{r_0}{2}\right)$, *the following estimate holds:*

$$m\left(f\overline{B}_r\right)\leqslant c_{p,\lambda}\,m\left(\overline{B}_r\right)\left(\log\frac{Q_0}{4m\left(\overline{B}_r\right)}\right)^{\frac{2}{2-p}}, \tag{10}$$

where $\overline{B}_r=\{z\in\mathbb{C}:|z|\leqslant r\}$ *and* $c_{p,\lambda}=\left(\frac{2}{\lambda}\right)^{\frac{2}{2-p}}\left(\frac{p-1}{2-p}\right)^{\frac{2(p-1)}{2-p}}$.

Power-logarithmic asymptotics

Theorem 2. *Let* $f:\mathbb{B}\to\mathbb{C}$ *be a ring* Q*-homeomorphism with respect to the* p*-modulus at a point* $z_0=0$, $1<p<2$, *and* $f(0)=0$. *Suppose, for some numbers* $\lambda>0$ *and* $r_0\in(0,1)$, *the condition*

$$Q_0=\int\limits_{B_{r_0}}\exp\left(\lambda Q(z)\right)dm(z)<\infty \tag{11}$$

is fulfilled. Then

$$\liminf_{z\to 0}\frac{|f(z)|}{|z|\left(\log\frac{1}{|z|}\right)^{\frac{1}{2-p}}}\leqslant k_{p,\lambda}<\infty, \tag{12}$$

where $k_{p,\lambda}=\left(\frac{4}{\lambda}\right)^{\frac{1}{2-p}}\left(\frac{p-1}{2-p}\right)^{\frac{p-1}{2-p}}$.

Example 1

Let $0 < \alpha < 1$ and $1 < p < 2$. Consider the mapping $f : \mathbb{B} \to \mathbb{C}$, where

$$f = \begin{cases} |z|^{\alpha-1} z, & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{13}$$

We show that f is a Q-homeomorphism with respect to the p-modulus with

$$Q(z) = \alpha^{1-p} |z|^{(\alpha-1)(2-p)}.$$

Next we find

$$f_z = \frac{(\alpha+1)\,|z|^{\alpha-1}}{2}, \quad f_{\overline{z}} = \frac{(\alpha-1)\,|z|^{\alpha-3} z^2}{2}.$$

Therefore,

$$l\left(f'(z)\right) = |f_z| - |f_{\overline{z}}| = \alpha\,|z|^{\alpha-1}$$

and

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = \alpha\,|z|^{2(\alpha-1)}.$$

Example 1

Now we calculate the inner p-dilatation using the formula

$$K_{I,p}(z,f) = \frac{J_f(z)}{l^p\left(f'(z)\right)} = \alpha^{1-p}|z|^{(\alpha-1)(2-p)}. \tag{14}$$

By Theorem 1.1 from [22], f is a Q-homeomorphism with respect to the p-modulus with $Q(z) = K_{I,p}(z,f) = \alpha^{1-p}|z|^{(\alpha-1)(2-p)}$.
It is easy to see that

$$\lim_{z\to 0} \frac{|f(z)|}{|z|\left(\log\frac{1}{|z|}\right)^{\frac{1}{2-p}}} = \lim_{z\to 0} \frac{|z|^{\alpha-1}}{\left(\log\frac{1}{|z|}\right)^{\frac{1}{2-p}}} = \infty \tag{15}$$

and, for some $r_0 \in (0,1)$ and every $\lambda > 0$,

$$\int\limits_{B_{r_0}} \exp\left(\lambda Q(z)\right) dm(z) = 2\pi \int\limits_0^{r_0} r \exp\left(\lambda \alpha^{1-p} r^{(\alpha-1)(2-p)}\right) dr = \infty.$$

[22] R. Salimov, E. Sevost'yanov. *The Poletskii and Vaisala inequalities for the mappings with* (p,q)*-distortion.* Complex Variables and Elliptic Equations. **59** (2014), no. 2, 217–231.

Example 2

Let $1 < p < 2$. Consider the mapping $f : \mathbb{B} \to \mathbb{C}$, where

(16)
$$f = \begin{cases} \left(\log \frac{e^{\frac{1}{2-p}}}{|z|} \right)^{\frac{1}{2-p}} z, \ z \neq 0, \\ 0, \ z = 0. \end{cases}$$

We write f in the polar coordinate system (r, θ):

$$f = \left(\log \frac{e^{\frac{1}{2-p}}}{r} \right)^{\frac{1}{2-p}} re^{i\theta}, \quad f(0) = 0.$$

We show that f is a Q-homeomorphism with respect to the p-modulus with

(17)
$$Q(z) = \left(\log \frac{e^{\frac{1}{2-p}}}{|z|} \right)^{p} \left(\log \frac{1}{|z|} \right)^{1-p}.$$

Example 2

Define

$$R(r) = r\left(\log\frac{e^{\frac{1}{2-p}}}{r}\right)^{\frac{1}{2-p}}. \tag{18}$$

Then

$$\lambda_1 = \frac{R(r)}{r} = \left(\log\frac{e^{\frac{1}{2-p}}}{r}\right)^{\frac{1}{2-p}}, \ \lambda_2 = R'(r) = \left(\log\frac{e^{\frac{1}{2-p}}}{r}\right)^{\frac{p-1}{2-p}} \log\frac{1}{r},$$

see Proposition 1.4 in [23]. It is obvious that $\lambda_2 < \lambda_1$. Therefore,

$$l\left(f'(z)\right) = \left(\log\frac{e^{\frac{1}{2-p}}}{|z|}\right)^{\frac{p-1}{2-p}} \log\frac{1}{|z|}$$

and

$$J_f(z) = \lambda_1\lambda_2 = \left(\log\frac{e^{\frac{1}{2-p}}}{r}\right)^{\frac{p}{2-p}} \log\frac{1}{r}.$$

[23] E. Sevost'yanov, *Mappings with Direct and Inverse Poletsky Inequalities.* Developments in Mathematics, vol. 78, Springer, Cham (2023)

Example 2

Hence,

$$K_{I,p}(z,f) = \frac{J_f(z)}{l^p\,(f'(z))} = \left(\log \frac{e^{\frac{1}{2-p}}}{|z|}\right)^p \left(\log \frac{1}{|z|}\right)^{1-p}. \tag{19}$$

Then, by Theorem 1.1 from [24], f is a Q-homeomorphism with respect to the p-modulus with

$$Q(z) = K_{I,p}(z,f) = \left(\log \frac{e^{\frac{1}{2-p}}}{|z|}\right)^p \left(\log \frac{1}{|z|}\right)^{1-p}. \tag{20}$$

It is easy to see that

$$\lim_{z\to 0} \frac{|f(z)|}{|z|\left(\log \frac{1}{|z|}\right)^{\frac{1}{2-p}}} = 1. \tag{21}$$

[24] R. Salimov, E. Sevost'yanov. *The Poletskii and Vaisala inequalities for the mappings with* (p,q)*-distortion.* Complex Variables and Elliptic Equations. **59** (2014), no. 2, 217–231.

Example 2

Next, for every $0 < \lambda < 2$ and some $r_0 \in \left(0, e^{-\delta_0}\right)$, where $\delta_0 = \frac{\lambda^{\frac{1}{p}}}{(2-p)\left(2^{\frac{1}{p}} - \lambda^{\frac{1}{p}}\right)}$, we obtain

(22)

$$\int\limits_{B_{r_0}} \exp(\lambda Q(z))dm(z) = 2\pi \int\limits_0^{r_0} r \exp\left(\lambda \left(\log \frac{e^{\frac{1}{2-p}}}{r}\right)^p \left(\log \frac{1}{r}\right)^{1-p}\right) dr$$

$$= 2\pi \int\limits_{\log \frac{1}{r_0}}^{\infty} \exp\left(-u\left(2 - \lambda\left(\frac{u + \frac{1}{2-p}}{u}\right)^p\right)\right) du.$$

Example 2

Since $u \geqslant \log\frac{1}{r_0}$,

$$\int_{\log\frac{1}{r_0}}^{\infty} \exp\left(-u\left(2-\lambda\left(\frac{u+\frac{1}{2-p}}{u}\right)^p\right)\right) du \leqslant \int_{\log\frac{1}{r_0}}^{\infty} e^{-cu} du = \frac{r_0^c}{c},$$

where $c = 2-\lambda\left(\frac{\log\frac{1}{r_0}+\frac{1}{2-p}}{\log\frac{1}{r_0}}\right)^p > 0.$

Therefore,

$$\int_{B_{r_0}} \exp(\lambda Q(z))\, dm(z) \leqslant \frac{2\pi r_0^c}{c} < \infty.$$

Thank you for your attention!