On locally nilpotent Lie algebras of derivations of small rank

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Let $\mathbb{K}$ be an arbitrary field of characteristic zero. Let A be an integral domain over $\mathbb{K}$ and R the fraction field over A. Denote by $\mathrm{Der}_{\mathbb{K}}A$ and $\mathrm{Der}_{\mathbb{K}}R$ the Lie algebras of $\mathbb{K}$-derivations of A and R, respectively. Any derivation $D \in \mathrm{Der}_{\mathbb{K}}A$ can be uniquely extended to a derivation of R. If $r \in R$ and $D \in \mathrm{Der}_{\mathbb{K}}A$, then one can define a $\mathbb{K}$-derivation $rD \in \mathrm{Der}_{\mathbb{K}}R$ by setting $rD(x) = r \cdot D(x)$ for all $x \in R$. We denote by $W(A)$ the subalgebra $R\mathrm{Der}_{\mathbb{K}}A = \mathbb{K}\langle rD \mid r \in R, D \in \mathrm{Der}_{\mathbb{K}}A\rangle$ of the Lie algebra $\mathrm{Der}_{\mathbb{K}}R$. For each subalgebra L of $W(A)$, we define the rank $\mathrm{rk}_R\, L$ of L over R as the dimension of the vector space $RL = \mathbb{K}\langle rD \mid r \in R, D \in L\rangle$ over R.

The structure of nilpotent subalgebras of finite rank over R from $W(A)$ was given in [2], [3], [4]. The results obtained there can be used to characterize locally nilpotent Lie algebras of derivations. A Lie algebra is called locally nilpotent if every its finitely generated subalgebra is nilpotent. As an example of such Lie algebras of derivations, one may consider the Lie algebras $u_n(\mathbb{K})$, $n \geq 1$, of triangular polynomial derivations (see, [1]).

Theorem 1. *Let L be a nonzero locally nilpotent subalgebra of finite rank over R from the Lie algebra $W(A)$. Let I be a proper ideal of L such that $I = RI \cap L$. Then the center $Z(L/I)$ of the Lie algebra L/I is nontrivial.*

In particular, we get the following result:

Corollary 2. *If L is a nonzero locally nilpotent subalgebra of finite rank over R from the Lie algebra $W(A)$, then the center $Z(L)$ of L is nonzero.*

Let L be a subalgebra of $W(A)$. Then the field of constants $F = F(L)$ for L consists of all $r \in R$ such that $D(r) = 0$ for all $D \in L$. In [2] it was proved that $FL = \mathbb{K}\langle fD | f \in F, D \in L\rangle$ is a Lie algebra over F. Moreover, if L is nilpotent and $\mathrm{rk}_R\, L < \infty$, then the Lie algebra FL is finite dimensional over F. In the case of a locally nilpotent L, FL is also locally nilpotent and the following theorem holds.

Theorem 3. *Let L be a locally nilpotent subalgebra of the Lie algebra $W(A)$. Let F be the field of constants for L. Then:*

(1) *If $\mathrm{rk}_R\, L = 1$, then L is abelian and $\dim_F FL = 1$;*
(2) *If $\mathrm{rk}_R\, L = 2$, then either FL is a nilpotent finite dimensional Lie algebra over F, or there exist $D_1,\ D_2 \in L$ and $a \in R$ such that*

$$FL = F\langle D_2, D_1, aD_1, \ldots, \frac{a^k}{k!}D_k, \ldots\rangle,$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$, and FL is infinite dimensional over F.

Rerefences

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