Growth of solutions of complex linear differential-difference equations with finite logarithmic order meromorphic coefficients

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In this work, we study the growth of meromorphic solutions of higher order complex linear differential- difference equations with meromorphic coefficients of finite logarithmic order. We use the fundamental results and the standard notations of the Nevanlinna value distribution theory for meromorphic functions [6, 7], such as the characteristic function $T(r,f)$, the proximity function $m(r,f)$ and the integrated counting function $N(r,f)$. In addition, we use the notations $\rho(f)$, $\tau(f)$ and $\lambda(\frac{1}{f})$ to denote respectively the order of growth of a meromorphic function f, the type of f and the exponent of convergence of the pole sequence of f. To express the rate of growth of a meromorphic function of zero order, we recall the following definitions.

Definition 1 ([3])**.** The logarithmic order of a meromorphic function f is defined by

$$\rho_{\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log T(r,f)}{\log\log r}.$$

When $1 \leq \rho_{\log}(f) = \rho < +\infty$, the logarithmic type of f is defined by

$$\tau_{\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{T(r,f)}{(\log r)^{\rho}}.$$

During the last decades, many papers have been devoted to the study of the growth of solutions of complex difference equations and complex differential equations by making use of Nevanlinna theory and its difference versions. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained independently by Hulburd-Korhonen [5] and Chiang-Feng [4]. As a generalization for both cases the linear difference equations and the linear differential equations, Zhou and Zheng considered in [8], the following linear differential-difference equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+c_i) = F(z), \tag{1}$$

where $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m \in \mathbb{N})$ and $F(z)$ are meromorphic functions of finite order, $c_i (i=0,\ldots,n)$ are distinct complex constants, and obtained the following result on the growth of solutions of the equation (1).

Theorem 1. *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m \in \mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l (0 \leq l \leq k)$ such that $A_{l0}(z)$ satisfies*

$$\lambda\left(\frac{1}{A_{l0}}\right) < \rho(A_{l0}) < \infty,$$

$$\max\{\rho(A_{ij})\colon (i,j)\neq(l,0)\}\leq\rho(A_{l0}),$$

$$\sum_{\rho(A_{ij})=\rho(A_{l0}),(i,j)\neq(l,0)}\tau(A_{ij})<\tau(A_{l0})<\infty.$$

1. *If* $\rho(F)<\rho(A_{l0})$*, or* $\rho(F)=\rho(A_{l0})$ *and* $\sum_{\rho(A_{ij})=\rho(A_{l0}),(i,j)\neq(l,0)}\tau(A_{ij})+\tau(F)<\tau(A_{l0})$*, or* $\rho(F)=\rho(A_{l0})$ *and* $\sum_{\rho(A_{ij})=\rho(A_{l0}),(i,j)=(l,0)}\tau(A_{ij})<\tau(F)$*, then every meromorphic solution* $f(z)(\not\equiv 0)$ *of (1) satisfies* $\rho(f)\geq\rho(A_{l0})$*. Further if* $F(z)\equiv 0$*, then* $\rho(f)\geq\rho(A_{l0})+1$*.*
2. *If* $\rho(F)>\rho(A_{l0})$*, then every meromorphic solution* $f(z)(\not\equiv 0)$ *of (1) satisfies* $\rho(f)\geq\rho(F)$*.*

Obviously, in Theorem 1 the case when the coefficients are of zero order is not included. Thus, a natural question arises: How we can express the growth of solutions of (1) for that case? The main purpose of this paper is to use the concepts of logarithmic order and the logarithmic type as growth indicators in order to answer the above question, such that we obtain some results on the logarithmic order under a main condition that there exists only one dominant coefficient by its logarithmic order or by its logarithmic type. Our results are the improvement and extensions of Theorem 1 and some previous results were obtained in [1] and [2].

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